Reaching Goals by a Deadline:
Digital Options and Continuous-Time Active Portfolio Management

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Abstract

We study a variety of optimal investment problems for objectives related to attaining goals by a fixed terminal time. We start by finding the policy that maximizes *the probability of reaching a given wealth level by a given fixed terminal time,* for the case where an investor can allocate his wealth at any time between $n + 1$ investment opportunities: n risky stocks, as well as a risk-free asset that has a positive return. This generalizes results recently obtained by Kulldorff and Heath for the case of a single investment opportunity. We then use this to solve related problems for the following cases: (i) when the investor has an external source of *income;* (ii) when the investor faces external *liabilities, as* arises in pension fund management; and (iii) when the investor is interested solely in beating the return of a given stochastic benchmark, as is sometimes the case in institutional money management. One of the benchmarks we consider for this last problem is that of the return of the *optimal growth policy,* for which the resulting controlled process is a supermartingale. Nevertheless, we still find an optimal strategy. For the general case, we provide a thorough analysis of the optimal strategy, and obtain new insights into the behavior of the optimal policy. For one special case, namely that of a single stock with constant coefficients, the optimal policy is independent of the underlying drift. We explain this by exhibiting a remarkable correspondence between the probability maximizing results and the pricing and hedging of a particular derivative security, known as a digital or binary option. In fact, we show that for this case, the *optimal policy to* maximize *the probability of reaching a* given value *of wealth by a predetermined time is equivalent to simply* buying *a European* digital *option with a particular strike price and payoff.* A similar result holds for the general case, but with the stock replaced by a particular (index) portfolio, namely the optimal *growth* or log-optimal portfolio.

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1 Introduction

There are various approaches to the problem of determining optimal dynamic investment policies, depending on the objectives of the investor. In continuous time, which is the setting in this paper, optimal dynamic investment policies for the objective of maximizing expected *utility* derived from terminal wealth or consumption over a finite horizon as well as discounted utility of consumption over. the infinite horizon are derived in the pioneering work of Merton [20]. Karatzas et al. [15], among others, consider generalizations that incorporate bankruptcies as well as more general price processes than those considered earlier. These results also have substantial implication for the pricing and hedging of contingent claims, see for example Karatzas [14] or Cox and Huang [6].

However, there are many investment scenarios where approaches alternative to that of utility maximization might be preferable. In particular, many actual investment objectives are related solely to the achievement of specific goals. For example, in institutional money management, the practice of *benchmarking is* quite prevalent. In this scenario, a portfolio manager is judged solely by how his portfolio performs relative to that of another benchmark portfolio, or index. The Standard and Poor's (S&P) 500 index is a typical example of a benchmark. There is a distinction made between passive *portfolio* management, and *active portfolio management (see e.g.,* Sharpe et al.[25]). A passive portfolio manager is simply interested in tracking the index, while an *active* portfolio manager is interested in beating the return of the predetermined given benchmark or index. From the viewpoint taken here, the passive portfolio manager's investment decision is uninteresting, since we assume that for all intents and purposes, a passive portfolio manager can simply invest directly in the benchmark. The active portfolio manager faces an interesting problem however, since he is investing in order to beat a "goal". The goal the active portfolio manager is trying to beat is the *stochastic* return of the benchmark. As another example, consider the a pension fund manager who is faced with a given liability stream, such as payouts to pensioners. The pension fund manager is interested in ensuring that all the liabilities are met by a given deadline. This is also a goal problem. In this paper we treat these two among other goal problems.

Optimal investment policies for objectives relating solely to the achievement of goals have been studied previously, although perhaps not to the extent that utility maximization has. For illustration, suppose the investor starts off with initial capital $a < X_o < b$. Then, some classical problems include determining an investment policy that (if appropriate) maximizes *the probability of reaching b before a,* or (if appropriate) minimizes *the expected time to the upper goal b.* We can refer to these, respectively, as the *survival problem,* and the *growth problem.* In discrete-time, and over an infinite horizon, the survival problem is the centerpiece of the classical work of Dubins and

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Savage [7], and the growth problem was first studied in Breiman [1] (as an asymptotic property of the policy first obtained in Kelly [17]). In continuous-time, the survival problem over an infinite horizon was first solved for diffusion processes in Pestien and Sudderth [23]. Survival problems related to various generalizations of the portfolio problem of Merton [20] were studied in Browne [3] and Browne [4], The growth problem in continuous-time was studied in a general framework in Heath et al. [12], Merton [21, Chapter 6], and generalized to a model that incorporated liabilities in Browne [4], Gottlieb [10] considered a particular model in which the two objectives were synthesized, specifically, minimizing the time to escape from an interval subject to a probability constraint on leaving the interval in the "right" direction. However, all these results are specific to the case of an *infinite horizon.* Since the performance of money managers is not judged over the infinite horizon, but rather over a finite (sometimes quite short) horizon, these studies are not directly applicable to the problem of active portfolio management. Similarly, in pension fund management, the horizon is typically finite. With a finite horizon, the distinction between a survival problem and a growth problem tends to blur, since in both cases they relate to maximizing probabilities: the survival problem would be to maximize the probability that the lower goal is not hit before the horizon, while the growth problem would be to maximize the probability that the upper goal *is* hit before the deadline.

A finite-horizon goal problem was studied recently in Kulldorff [19], for a model with a single risky favorable investment opportunist y. Kulldorff [19] obtained the optimal investment policy for the objective of maximizing *the probability that wealth* attains a given constant goal *by a fixed terminal time.* The return of the single risky asset was modeled as a Brownian motion with a time-dependent drift coefficient and a constant diffusion coefficient. The goal as well as the constant diffusion coefficient were both normalized to one. Heath [11] considered the same model, but with constant drift as well and gave a different proof of the results in Kulldorff [19]. In both cases, there was no risk-free asset available other than cash, which had a zero return. One of the interesting features of their policy was the fact that for the case of a constant drift, the optimal policy was *independent of this* underlying drift. No explanation of this rather remarkable fact was given.

In this paper, we address a variety of more general goal problems, all with probability maximizing objectives. To that end we first generalize the important results of Kulldorff [19] and Heath [11] in a few fundamental ways: First, we expand the investment opportunity set to include a risk-free asset that has a positive time-dependent rate of return, as well as multiple risky assets with time-varying covariance structure. Our derivation and proof of the resulting optimal policy differs considerably from, and simplifies, that of Kulldorff [19] – we first obtain a candidate optimal value function by applying an extended version of the elegant argument in Heath [11], and then apply

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a new verification argument. We then extend our general results to treat the following cases: (i) where the investor earns income from an exogenous source; (ii) where the investor has an external *liability* stream, or forced consumption, as is the case in pension fund management; and (iii) where the investor's objective is to beat the (stochastic) return of a given benchmark portfolio.

The explanation as to why the optimal policy is independent of the underlying drift for the single stock case with constant coefficients is provided as a byproduct of our analysis for the more general case. It turns out that when there are multiple risky stocks, the optimal policy, for maximizing the probability of attaining a preset level of wealth by a finite deadline, is no longer independent of the drift parameters, even for *the case of constant coefficients*. The resulting policy is quite interesting and we provide a new analysis that allows for a complete quantitative assessment of the risk-taking behavior of an investor following such an objective. Furthermore, we obtain a new representation of the optimal wealth process. This represent at ion, together with the addition of a risk-free asset, allows us to exhibit a remarkable correspondence between the probability maximizing policy and the hedging strategy of a digital option for the single-stock constant coefficients case. In particular we show that the optimal dynamic investment strategy for the objective of maximizing the probability of reaching a given goal by a fixed terminal time is *completely equivalent to the (static) investment strategy which simply* purchases a *European digital call* option on *the underlying stock, with a particular strike price and payoff.* This result is of independent interest since it provides an example where a policy which is optimal for an objective stated on *wealth,* is equivalent to the purchase of an option on the underlying *stock.* Moreover, it also implicitly contains the explanation as to why the optimal policy is independent of the drift in the constant coefficients single-stock case. For the general case, we are able to show a similar result, however with the single stock replaced by the return of a particular portfolio policy: the *optimal growth,* or *log-optimal* portfolio. Specifically, we will show that the probability maximizing policy is completely equivlaent to purchasing a European digital option on the return of the log-optimal portfolio.

A summary and outline of the remainder of the paper is as follows: In the next section, we introduce the basic model with multiple stocks and a risk-free asset with positive return. In Section 3 we provide the optimal policy for the problem of maximizing the probability of reaching the goal by terminal time $T < \infty$, as well as the new representation of the optimal wealth under this policy (Corollary 3.2 below). The proof is delayed until Section 6. In Section 4, we then use this representation to show the correspondence between the single stock case with constant coefficients and a digital option on the stock, as well as the correspondence with the digital option on the log-optimal portfolio in the general case,

In Section 5, we analyze the optimal policy for the general case. We first show that the optimal policy can be interpreted as a linear function of wealth, where the coefficient decomposes into the product of two distinct factors: (1) a purely *time-dependent risk factor,* which is determined solely by the risk premiums of the stocks and the time remaining until the deadline; and (2) a purely *state-dependent* function which is parameterized solely by the current *percentage of the distance to the* (in our case, time-dependent) *goal achieved.* As intuition would suggest, the former function typically increases as the horizon decreases, while the latter function decreases as the percentage increases. The optimal policy is therefore a dynamic portfolio strategy that continuously rebalances the portfolio weights depending upon how much time remains to the deadline as well as how close the current wealth is to the goal. The interplay between these two factors is analyzed to a fairly explicit extent next when we analyze the region where borrowing takes place. It turns out that this region is determined by a single equation involving the "risk-adjusted" remaining time, and the percentage of the goal achieved to that point.

In Section 7 we consider the case where the investor earns income from an external source other than trading gains. We show that contrary to utility maximizing strategies – where an investor uses the exogenous income to take a more risky position in stocks than he would otherwise – a "probability maximizing" investor relies on this exogenous income to be more cautious. In particular, we show that external income causes the investor to incorporate a performance bound: if the performance of the stocks is such that wealth ever falls to the level that could have been achieved by simply investing all the previous income into the risk-free asset, then all investment in the risky stocks ceases.

In Section 8, we consider the case where the investor has a given liability stream that must be met, as is the case in pension fund management. In this case it is possible for wealth to become negative due to the external liabilities. We find the optimal policy for maximizing the probability of reaching a preset level of wealth, subject to the regulatory constraint that reserves are always greater than the discounted value of the future liabilities.

In Section 9, we consider the case where the investor's goal is to beat the return of a given (stochastic) benchmark by a prespecified amount by a predetermined time. We also find the related policy that allows the investor to control for the downside risk. When the stochastic benchmark is given by the optimal *growth policy,* or equivalently, the policy that maximizes logarithmic utility, which is sometimes referred to as the *market portfolio* in continuous-time finance, then certain complications arise. Specifically, it is well known that the ratio of the return from any arbitrary portfolio strategy to the return generated by the optimal growth strategy is a nonnegative local martingale, hence a supermartingale (see, e.g., Karatzas [14, Section 9.6]). Nevertheless, we find

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a policy that does achieve the theoretical upper bound on the probability of beating the return of the optimal growth by a predetermined amount, and is hence an optimal policy.

2 The Model

The model under consideration here is that of a complete market as in Merton [20, 21], Karatzas [14] and others, wherein there are n (correlated) risky assets generated by n independent Brownian motions. The prices of these stocks are assumed to evolve as

$$dS_i(t) = S_i(t) \left[\mu_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t)dW_t^{(j)} \right], i = 1, \ldots, n, \tag{2.1}$$

where $\mu_i(t), \sigma_{ij}(t)$ are deterministic functions, for $i, j, = 1, \ldots, n$, and $t \geq 0$, and where $\boldsymbol{W}_t := (W_t^{(1)}, \ldots, (W_t^{(n)})'$ is a standard n-dimensional Brownian motion, defined on the complete probability space $(\Omega, \mathcal{F}, P)$, where $\{\mathcal{F}_t, t \geq 0\}$ is the P-augmentation of the natural filtration $\mathcal{F}_t^{\boldsymbol{W}} = \sigma\{\boldsymbol{W}_s; 0 \leq s \leq t\}$. Thus the stock prices follow a *correlated* time-dependent geometric Brownian motion in $\mathbf{R}^n$.

There is also a riskless asset whose price, B_t, evolves according to

$$dB_t = r(t)B_t dt. \tag{2.2}$$

Let $\{\boldsymbol{f}_t, 0 \leq t \leq T\}$ denote a vector control process where $\boldsymbol{f}_t = \left(f_t^{(1)}, \ldots, f_t^{(n)}\right)'$. The interpretation of $f_t^{(i)}$ is the *total amount of money invested in the i-th stock at time t*. It is assumed that $\boldsymbol{f}_t$ is admissible, in that $\boldsymbol{f}_t$ is a nonanticipating, $\mathcal{F}_t$-adapted process that satisfies $\int_0^t \boldsymbol{f}_s' \boldsymbol{f}_s ds < \infty$ for all t.

Let X_t^f denote the *wealth* of the investor at time t, under an investment policy $\{\boldsymbol{f}_t\}$, with $X_0 = x$. Since any amount not invested in the risky stock is held in the bond, this wealth process then evolves as

$$\begin{aligned} dX_t^f &= \sum_{i=1}^{n} f_t^{(i)} \frac{dS_i(t)}{S_i(t)} + \left(X_t^f - \sum_{i=1}^{n} f_t^{(i)} \right) \frac{dB_t}{B_t} \\ &= \left[r(t)X_t + \sum_{i=1}^{n} f_t^{(i)}(\mu_i(t) - r(t)) \right] dt + \sum_{i=1}^{n} \sum_{j=1}^{n} f_t^{(i)} \sigma_{ij}(t) dW_t^{(j)}, \end{aligned} \tag{2.3}$$

upon substituting from (2.1) and (2.2).

If we introduce now the matrix functions $\boldsymbol{\sigma}(t) = (\sigma(t))_{ij}$, and the (column) vectors $\boldsymbol{\mu}(t) = (\mu_1(t), \ldots, \mu_n(t))'$, $\mathbf{1} = (1, \ldots, 1)'$, and then set $\boldsymbol{\Sigma}(t) = \boldsymbol{\sigma}(t)\boldsymbol{\sigma}(t)'$, we may rewrite (2.3) as

$$dX_t^f = \left[r(t)X_t^f + \boldsymbol{f}_t' \left(\boldsymbol{\mu}(t) - r(t)\mathbf{1} \right) \right] dt + \boldsymbol{f}_t' \boldsymbol{\sigma}(t) d\boldsymbol{W}_t. \tag{2.4}$$

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For Markov control processes $\{f_t\}$ and functions $\Psi(t,x) \in \mathcal{C}^{1,2}$, we may therefore write the generator of the (one-dimensional) wealth process as

$$\mathcal{A}^f \Psi(t,x) = \Psi_t + \left(r(t)x + f_t'(\mu(t) - r(t)\mathbf{1}) \right) \Psi_x + \frac{1}{2} f_t' \Sigma(t) f_t \Psi_{xx}. \tag{2.5}$$

The assumption of completeness implies that $\sigma^{-1}(t)$, and hence, $\Sigma^{-1}(t)$ exists.

The fundamental vector O(t), defined by

$$\theta(t) := \sigma^{-1}(t)(\mu(t) - r(t)\mathbf{1}) \tag{2.6}$$

plays a pivotal role here since then according to the Girsanov theorem (cf. [16, Section 3.3.5]) the vector **process**

$$\tilde{W}_t = W_t + \int_0^t \theta(s)ds \tag{2.7}$$

is an n dimensional standard Brownian motion under the measure $\tilde{P}$, where $\tilde{P}$ is the measure defined by

$$\frac{dP(\omega)}{d\tilde{P}(\omega)} = \exp\left\{ \int_0^T \theta(s)'dW_s + \frac{1}{2}\int_0^T \theta(s)'\theta(s)\,ds \right\} \equiv \exp\left\{ \int_0^T \theta(s)'d\tilde{W}_s - \frac{1}{2}\int_0^T \theta(s)'\theta(s)\,ds \right\}. \tag{2.8}$$

The measure $\tilde{P}$ is sometimes referred to as the *risk-neutral* (or equivalent martingale) measure, and the vector $\theta(t)$ is also called the vector of *risk premiums,* or the market price of risk.

In the next section, we will give the optimal policy for the problem of maximizing the probability that terminal wealth exceeds a predetermined level at the predetermined time T. The proof of this theorem will be deferred until later, in Section 6.

3 Maximizing the probability of reaching a goal in finite-time

In this section, we present the optimal value function and optimal investment policy for maximizing the *probability* that terminal wealth at time T exceeds a given threshold b, with $X_0 < b$.

Let $P_{(t,x)}(\cdot) := P\left(\cdot \,|X_t^f = x \right)$, and for any admissible policy f, let $V^f(t,x;b) := P_{(t,x)}\left(X_T^f \geq b \right)$. In the following theorem, we give the optimal value function $V(t,x;b) := \sup_{f \in \mathcal{G}} V^f(t,x;b)$ as well as the associated optimal control policy $f_t^* = \arg\sup_{f \in \mathcal{G}} V^f(t,x;b)$, where $\mathcal{G}$ denotes the set of admissible controls, for an investor whose wealth process evolves according to (2.3).

Remark on notation: For the sequel, $\Phi(\cdot)$ is the CDF of a standard normal variate, and $\phi(\cdot)$ is the associated PDF.

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Theorem 3.1 *For $\{X_t^f, 0 \le t \le T\}$ given by (2.3), let $V(t,x;b) = \sup_{f \in \mathcal{G}} P_{(t,x)}\left(X_T^f \ge b\right)$, with optimal control vector f_t^*. Then*

$$V(t,x;b) = \Phi\left(\Phi^{-1}\left(\frac{x}{b}e^{\int_t^T r(s)ds}\right) + \sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,ds}\right) \tag{3.1}$$

and the optimal policy is

$$f_t^*(x;b) = \left[\frac{\sigma^{-1}(t)'\boldsymbol{\theta}(t)}{\sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,ds}}\right] be^{-\int_t^T r(s)ds}\,\phi\left(\Phi^{-1}\left(\frac{x}{b}e^{\int_t^T r(s)ds}\right)\right). \tag{3.2}$$

Remark 3.1: Since $\Phi^{-1}(u) = \infty$ for $u \ge 1$, (3.1) shows that for $x > be^{-\int_t^T r(s)ds}$, we would have $V(t,x;b) = 1$, and correspondingly, (3.2) shows that we would in that case have $f_t^* = \mathbf{0}$, where $\mathbf{0} = (0,0,\dots,0)'$, whereby all the investor's wealth will be invested in the riskless asset.

This is due, of course, to the fact that when there is a risk-free asset available for investment that has a positive return of $r(t)$ per unit time, the problem of reaching the goal b at time T from the state (t,x) (i.e., from wealth $X_t = x$) is nontrivial and interesting *only for the case where* $x < be^{-\int_t^T r(s)ds}$, for if the converse holds, i.e., $x \ge be^{-\int_t^T r(s)ds}$, then the simple strategy of placing all the current (time t) wealth into the bond will yield a terminal wealth of $xe^{\int_t^T r(s)ds}$, which would in turn beat the goal b with probability one. Thus ensuring that terminal wealth at time T exceeds the fixed level b is equivalent to ensuring that for some $t < T$, wealth exceeds the time-dependent level $be^{-\int_t^T r(s)ds}$, i.e.,

$$P\left(\sup_{0 \le s \le T} X_s^f \ge b\right) \equiv P\left(X_t^f \ge be^{-\int_t^T r(s)ds},\ \text{for some } t \le T\right).$$

If in Theorem 3.1 we take $n = 1, b = 1$, as well as $r(s) = 0$, and $\sigma(s) = 1$ for all s, then we recover the results of Kulldorff [19, Theorem 7]. Heath [11] used a different methodology to treat the *constant coefficients* version of the problem, with $\mu(s) = \mu$, $r = 0$.

Remark 3.2: Constant Coefficients. For the case of constant coefficients, i.e., when $\sigma(t) = \sigma, \mu(t) = \mu$, and $r(t) = r$, for all t, the results of Theorem 3.1 reduce to

$$V(t,x;b) = \Phi\left(\Phi^{-1}\left(\frac{x}{b}e^{r(T-t)}\right) + \sqrt{\boldsymbol{\theta}'\boldsymbol{\theta}(T-t)}\right) \tag{3.3}$$

and the optimal policy is

$$f_t^*(x;b) = \left[\frac{\Sigma^{-1}(\mu - r\mathbf{1})}{\sqrt{\boldsymbol{\theta}'\boldsymbol{\theta}(T-t)}}\right] be^{-r(T-t)}\,\phi\left(\Phi^{-1}\left(\frac{x}{b}e^{r(T-t)}\right)\right) \tag{3.4}$$

where $\theta = \sigma^{-1}(\mu - r\mathbf{1})$ is a constant vector. Note that for this case, when $n = 1$ (and so there is only one risky stock), then $\theta = \frac{\mu-r}{\sigma}$, and the optimal control f_t^* reduces to

$$f_t^*(x;b) = \left[\frac{1}{\sigma\sqrt{T-t}}\right]be^{-r(T-t)}\phi\left(\Phi^{-1}\left(\frac{x}{b}e^{r(T-t)}\right)\right).\qquad(3.5)$$

It is important to note that it is only for this single-stock case that the optimal control is in fact *independent of the* underlying drift *parameter as* is apparent from (3.5). In this case, the investor always invests less in the risky stock when there is a risk-free asset with r > 0 than he would in the corresponding case with r = O, treated earlier in [19] and [11]. This follows directly from the fact that for any $a > 0$, $e^{-a}\phi(ze^a) < \phi(z)$, for any z. (In the general case, this may or may not be true, depending on the relationship between $r(s), \mu(s)$ and $\sigma(s)$.) As we will show in the next section, the policy of (3.5) is intimately connected to the *hedging* strategy for a particular type of derivative security known as a *digital option.*

3.1 The Optimal Wealth Process

When the control function f_t^* is placed back into the evolutionary wealth equation (2.4), we obtain an optimal wealth process, $\{X_t^*, 0 \le t \le T\}$, that satisfies the stochastic differential equation

$$
\begin{aligned}
dX_t^* &= \left[r(t)X_t^* + \frac{\theta(t)'\theta(t)}{\sqrt{\int_t^T \theta(s)'\theta(s)ds}}be^{-\int_t^T r(s)ds}\phi\left(\Phi^{-1}\left(X_t^*\left[be^{-\int_t^T r(s)ds}\right]^{-1}\right)\right)\right]dt \\
&\quad + \frac{be^{-\int_t^T r(s)ds}}{\sqrt{\int_t^T \theta(s)'\theta(s)ds}}\phi\left(\Phi^{-1}\left(X_t^*\left[be^{-\int_t^T r(s)ds}\right]^{-1}\right)\right)\theta(t)'dW_t, \quad \text{for } t \le T. \quad(3.6)
\end{aligned}
$$

We will show later that the solution to this stochastic differential equation is given by the following corollary.

Corollary 3.2 *The optimal wealth process, X_t^*, for $0 \le t \le T$, is given by*

$$X_t^* = be^{-\int_t^T r(s)ds}\Phi\left(\frac{\int_0^t \theta(s)'dW_s + \int_0^t \theta(s)'\theta(s)ds + \sqrt{\int_0^T \theta(s)'\theta(s)ds}\ \Phi^{-1}\left(\frac{X_0}{b}e^{\int_0^T r(s)ds}\right)}{\sqrt{\int_t^T \theta(s)'\theta(s)ds}}\right).$$

$$(3.7)$$

This representation provides the link between the probability maximizing objective and the digital, or *binary* option, and contains the explanation as to why it is only in the single stock constant coefficient case that the policy is independent of the underlying dirft. We discuss this directly in the next section. After that, we return to analyze the optimal policy in explicit detail. The proof of Theorem 3.1 will then be provided in the following section. After that we examine cases that include income as well as liabilities, and conclude with the the problem of beating an index.

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4 Connections with Digital Options

4.1 Constant Coefficients

Consider a Black-Scholes [2] world with a single stock whose price, St, follows the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{4.1}$$

as well as a risk-free asset with constant return r.

A *digital (or binary) option* on this stock, with strike price *K* and payoff *B,* is a contract that pays \$B at time T if $S_T \geq K$. Thus a digital option amounts to a straight bet on the terminal price of the underlying stock, This is a particular example of an 'exotic' option that is discussed in most standard references on options, such as Hull [13] or Wilmott et al. [27]. Digital options are currently traded over the counter.

Let *C(t, St)* denote the current rational price of such an option. Then, a standard Black-Scholes pricing argument shows that

$$C(t, S_t) = Be^{-r(T-t)} \Phi \left(\frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \right). \tag{4.2}$$

The underwriter of such an option (i.e., the party that agrees to pay \$B at T if $S_T > K$) is interested in *hedging* its risk. A dynamic *hedging strategy* for the writer of such an option is a dynamic investment policy, say $\{\Delta_t, t \leq T\}$, which holds Δ_t shares of the underlying stock at time *t* so as to ensure that the underwriter's position is riskless at all times. It is also well known that this hedging, or replicating, strategy is given by $\Delta_t = C_2(t, S_t)$, where $C_2(t, x) = \frac{\partial C}{\partial x}$. It is easy to see that the hedging strategy for the digital option is simply

$$\Delta_t = Be^{-r(T-t)} \phi \left(\frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \right) \frac{1}{S_t \sigma\sqrt{T-t}}. \tag{4.3}$$

Observe that since At is the *number of shares* of the underlying stock the investor holds at time *t,* the actual amount of money invested in the stock at time *t* is $\Delta_t \cdot S_t$.

General treatment of options that discuss pricing and hedging of various options, including the digital and the derivation of (4.2) and (4.3) can be found in such basic texts as [13] and [27]. A valuable source for more theoretical issues is the survey by Karatzas [14].

To see the connection with our problem, consider an investor who at time *t* has sold this digital option for the Black-Scholes price of *C(t, St),* and suppose the investor will then invest the proceeds in such a manner as to *maximize the probability that he can pay off the claim of this option at time*

T, i.e., for all intents and purposes, the investor's "wealth" at time *t* is *C(t,St),* and the investor will then invest this wealth so as to maximize *the probability that the terminal* fortune *from this' strategy is equal to B.* Our previous results show that the optimal policy is at time *t* is given by f_t^* of (3.5) with $x = C(t, S_t)$ and $b = B$, i.e., by $f_t^*(C(t, S_t); B)$. But placing $C(t, S_t)$ of (4.2) into (3.5) with $b = B$ and simplifying gives

$$
\begin{aligned}
f_t^*(C(t, S_t); B) &= Be^{-r(T-t)} \phi \left(\Phi^{-1} \left(\frac{C(t, S_t)}{B} e^{r(T-t)} \right) \right) \frac{1}{\sigma\sqrt{T-t}} \\
&= Be^{-r(T-t)} \phi \left(\frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \right) \frac{1}{\sigma\sqrt{T-t}} \\
&\equiv \Delta_t \cdot S_t
\end{aligned}
\tag{4.4}
$$

where Δ_t is given by (4.3). Thus, in this case, f_t^* is equivalent to the hedging strategy of the digital option.

Moreover, if we specialize the representation of the optimal wealth process given in Corollary 3.2, i.e. X_t^* of (3.7), to the single stock case with constant coefficients, we find that

$$
X_t^* = Be^{-r(T-t)} \Phi \left(\frac{\sigma W_t + (\mu - r)t + \sigma\sqrt{T}\Phi^{-1}\left(X_0 e^{rT}/B\right)}{\sigma\sqrt{T-t}} \right)
\tag{4.5}
$$

is the wealth of an investor at time *t* who started off with initial wealth X_0 and is investing so as to maximize the probability of his terminal time *T* wealth being equal to *B*.

Observe now that by (4.1) we have

$$
S_t = S_0 \cdot \exp\left\{ \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t \right\}
$$

from which we may infer that

$$
\sigma W_t + (\mu - r)t \equiv \ln(S_t/S_0) - \left(r - \frac{1}{2}\sigma^2\right)t.
$$

When this is placed back into (4.5), it yields the following representation of the optimal wealth process in terms of the underlying stock price

$$
X_t^* = Be^{-r(T-t)} \Phi \left(\frac{\ln(S_t/S_0) - \left(r - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{T}\ \Phi^{-1}\left(X_0 e^{rT}/B\right)}{\sigma\sqrt{T-t}} \right).
\tag{4.6}
$$

Note that there is no explicit dependence on in (4.6). If the investor's initial wealth is taken to be the Black-Scholes price of the digital option at time 0, i.e., take $X_0 \equiv C(0, S_0)$, where $C(\cdot, \cdot)$ is given by (4.2), then it is seen that (4.6) reduces to

$$
X_t^* \equiv C(t, S_t)
$$

where $C(t, S_t)$ is given by (4.2), i.e., *the optimal wealth process under policy f_t^* is just the Black-Scholes price for a digital option with payoff B !*

It is interesting to note that the Black-Scholes value (4.2) and its resulting hedging strategy (4.3) are both calculated and determined by the *risk-neutral* probability measure (under which is replaced by r), while the optimal strategy for maximizing the probability y of terminal wealth being greater than *B* was determined under the regular measure.

The analysis above can be inverted to show the following rather interesting fact:

Proposition 4.1 *Consider an investor, with initial wealth X_o, whose objective is to maximize the probability that terminal wealth at* time *T exceed some fixed level B, and who has the following two investment opportunities — the risky stock St of (4.1)* and a *bond with constant return r. Then investing according to the* dynamically *optimal policy $\{f_t^*, 0 \le t \le T\}$ of (3.5) is equivalent to the* static policy *that* purchases *(at time O) one European call digital option with payoff B and strike price K*, where K* is given by*

$$K^* = S_0 \cdot \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)T - \sigma\sqrt{T}\,\Phi^{-1}\left(X_0 e^{rT}/B\right)\right\}.\tag{4.7}$$

Proof: The representation of the optimal wealth process under f_t^*, i.e., X_t^* of (4.6), shows that, for a fixed X_0, the event $\{X_T^* \ge B\}$ is equivalent to the event $\{S_T \ge K^*\}$.   ∎

Note that (4.7) is equivalent to

$$X_0 = Be^{-rT}\Phi\left(\frac{\ln\left(\frac{S_0}{K^*}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right),$$

which when compared with (4.2) shows that X_0 is simply the Black-Scholes price at time $t = 0$ of a digital option with payoff B and strike price K^*.

Remark 4.1: It is important to note that the results of this section are specific to the constant coefficient case in one-dimension and do not generalize to the time-dependent case in even one-dimension, nor to the multi-dimensional constant-coefficient case. The reason for this is that it is only for the constant coefficients one-dimensional case that the representation of X_t^* of (3.7) reduces to the form in (4.6), whereby we see that X_t^* is determined completely by the *current* value of the underlying stock. As such, we can represent the event $\{X_T^* \ge B\}$ in terms of a corresponding simple event determined by S_T alone, which leads to the equivalence with the digital option, whose price must be independent of μ by risk neutral valuation arguments. For more general cases, this does not occur. However, as we will show directly, the general case is intimately connected to a digital option on a particular portfolio, namely the *growth-optimal portfolio.*

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4.2 The General Case: Optimal Growth and Digital Options

For the general model considered earlier, let π_t^* denote the vector

$$\pi_t^* := \Sigma^{-1}(t)\left[\mu(t) - r(t)\mathbf{1}\right] \equiv \sigma^{-1}(t)'\theta(t). \tag{4.8}$$

The corresponding specific portfolio policy $\{f_t^{\pi^*}\}$, where $f_t^{\pi^*}(x) = \pi_t^* \cdot x$, is the policy that maximizes *logarithmic* utility of wealth and also maximizes the *growth rate* of wealth (cf. Merton [21, Chapter 6], Karatzas [14, Section 9.6], Browne and Whitt [5]). For the case of constant coefficients, it is also optimal for *minimizing the expected time to reach any specific level of wealth* (in an infinite horizon setting) (cf. Heath et al. [12], Merton [21], Browne [4]). As such, $\{\pi_t^*\}$ is referred to as the *optimal growth policy*. It is also sometimes referred to as the *market portfolio* (from CAPM considerations).

Let Π_t denote the corresponding "wealth" under policy π_t^*. Then taking $f_t = \pi_t \cdot X_t^f$ and $X_t^f = \Pi_t$ in (2.4) shows that Π_t satisfies the linear stochastic differential equation

$$d\Pi_t = \Pi_t\left[r(t) + \theta(t)'\theta(t)\right]dt + \Pi_t\theta(t)'dW_t$$

and as such we have

$$\Pi_t = \Pi_0 \exp\left\{\int_0^t \left[r(s) + \frac{1}{2}\theta(s)'\theta(s)\right]ds + \int_0^t \theta(s)'dW_s\right\}. \tag{4.9}$$

We can also represent the wealth process under the optimal growth policy in terms of the change-of-measure of (2.8) by

$$\Pi_t = \Pi_0 e^{\int_0^t r(s)ds}E\left(\frac{dP}{d\tilde{P}}|\mathcal{F}_t\right).$$

The main result of this section is that the optimal policy of Theorem 3.2 is equivalent to the hedging strategy for a digital option on π_T, and so the optimal wealth process of Corollary 3.2 is therefore equivalent to the Black-Scholes price on this option. Before we state this formally, recognize that for every $t \geq 0$, the optimal growth portfolio, Π_t, is equivalent in distribution to the process $\hat{\Pi}_t$, where

$$d\hat{\Pi}_t = \hat{\Pi}_t\left[r(t) + \sigma_*^2(t)\right]dt + \hat{\Pi}_t\sqrt{\sigma_*^2(t)}\,d\hat{W}_t \tag{4.10}$$

where $\hat{W}_t$ is an independent Brownian motion (in $\mathbf{R}^1$), and $\sigma_*^2(t)$ is defined by

$$\sigma_*^2(t) := \theta(t)'\theta(t) \equiv \pi_t^{*'}\Sigma(t)\pi_t^*. \tag{4.11}$$

As such, $\sigma_*^2(t)$ is the volatility of the the optimal growth portfolio.

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Remark 4.2: Note that in terms of the optimal growth policy, we may write the optimal policy, $\boldsymbol{f}_t^*$ of (3.2) in Theorem 3.1 as

$$\boldsymbol{f}_t^*(x;b) = \boldsymbol{\pi}_t^* \frac{1}{\sqrt{\int_t^T \sigma_*^2(s)\,ds}} \, be^{-\int_t^T r(s)ds}\,\phi\left(\Phi^{-1}\left(\frac{x}{b}e^{\int_t^T r(s)ds}\right)\right).\qquad (4.12)$$

Proposition 4.2 *Consider an investor whose wealth, $\{X_t^f\}$, satisfies (2.4), and whose objective is to maximize $P\left(X_T^f \geq b\right)$. Then the optimal policy for this objective, given by $\boldsymbol{f}_t^*$ of (3.2) in Theorem 3.1, is completely equivalent to the* static *policy which purchases a* European *digital option on Π_T with payoff b and strike price K^{**}, where*

$$K^{**} = \Pi_0 \exp\left\{\int_0^T \left[r(s) - \frac{\sigma_*^2(s)}{2}\right]ds - \sqrt{\int_0^T \sigma_*^2(s)ds}\ \Phi^{-1}\left(\frac{X_0 e^{\int_0^T r(s)ds}}{b}\right)\right\}.\qquad (4.13)$$

Equivalently, under the optimal policy $\boldsymbol{f}_t^*$, the optimal wealth $\{X_t^*\}$ is equivalent to the (no arbitrage) Black-Scholes price of this digital option on Π_T, i.e.,

$$X_t^* = be^{-\int_t^T r(s)ds}\Phi\left(\frac{\ln\left(\frac{\Pi_t}{K^{**}}\right) + \int_t^T \left[r(s) - \frac{1}{2}\sigma_*^2(s)\right]ds}{\sqrt{\int_t^T \sigma_*^2(s)ds}}\right),\quad \text{for all } t \leq T.\qquad (4.14)$$

Remark 4.2: Observe from (4.14) that terminal wealth, X_T^* is either 0 or b, with

$$\{X_T^* = b\} \Leftrightarrow \{\Pi_T \geq K^{**}\}.$$

Proof: We can use (4.9) to write the representation of the optimal wealth X_t^* of (3.7) in terms of the optimal growth portfolio, Π_t, (using $\sigma_*^2(t)$) as

$$X_t^* = be^{-\int_t^T r(s)ds}\Phi\left(\frac{\ln\left(\frac{\Pi_t}{\Pi_0}\right) - \int_0^t \left[r(s) - \frac{1}{2}\sigma_*^2(s)\right]ds + \sqrt{\int_0^T \sigma_*^2(s)ds}\ \Phi^{-1}\left(\frac{X_0 e^{\int_0^T r(s)ds}}{b}\right)}{\sqrt{\int_t^T \sigma_*^2(s)ds}}\right).$$

When we substitute K^{**} of (4.13) back into this, we obtain (4.14).

It remains to show that (4.14) is indeed the Black-Scholes price of a digital option on Π_T. To see this, recall that if $\psi(\Pi_T)$ is the payoff on any contingent claim written on Π_T, then the (no arbitrage) Black-Scholes price of this claim at time t is given by $C(t,\Pi_t)$, where

$$C(t,\Pi_t) = \tilde{E}\left[e^{-\int_t^T r(s)ds}\psi(\Pi_T)|\mathcal{F}_t\right],\qquad (4.15)$$

where $\tilde{E}$ denotes the expectation taken under the risk-neutral measure $\tilde{P}$ of (2.8).

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Observe now that in terms of the Brownian motion $\{\tilde{W}_t\}$ of (2.7), we may write Π_t of (4.9) as

$$\Pi_t = \Pi_0 \exp\left\{ \int_0^t \left[r(s) - \frac{1}{2}\boldsymbol{\theta}(s)'\boldsymbol{\theta}(s) \right] ds + \int_0^t \boldsymbol{\theta}(s)'d\tilde{W}_s \right\}, \tag{4.16}$$

and so if – for an arbitrary strike price K – we take $\psi(\Pi_T) = b \cdot 1\{\Pi_T \geq K\}$, then a simple computation shows that for this case we have

$$C(t,\Pi_t) = be^{-\int_t^T r(s)ds} \Phi\left(\frac{\ln\left(\frac{\Pi_t}{K}\right) + \int_t^T \left[r(s) - \frac{1}{2}\sigma_*^2(s) \right] ds}{\sqrt{\int_t^T \sigma_*^2(s)ds}} \right), \tag{4.17}$$

from which it is apparent that (4.14) is indeed the (time t) Black-Scholes price of a digital option on Π_T, with payoff b and strike price K^{**}. ∎

We now move on to analyze the optimal investment policy, $\boldsymbol{f}_t^*$ of Theorem 3.1 for the general case in explicit detail.

5   Analysis of the optimal policy in the general case

In this section we examine the investment policy $\boldsymbol{f}_t^*$ of (3.2) for the general case in greater detail. Note that for a fixed $t < T$, as wealth, x, gets closer to the "goal" $be^{-\int_t^T r(s)ds}$, we have $\boldsymbol{f}_t^* \to \mathbf{0}$. Similarly, as the wealth gets closer to the barrier 0, we also have $\boldsymbol{f}_t^* \to \mathbf{0}$. Thus if wealth is close to 0 with enough time remaining on the clock, the investor does not "panic" and start taking aggressive positions in the stocks to get away from 0, but rather the investor should be patient and wait for his wealth to grow a bit before taking active positions in the risky stocks. (It is interesting to note that if there is enough time remaining until the deadline, the investor does not even borrow an excessive amount when his wealth is close to 0, even though he does have the ability to borrow an unlimited amount. See the discussion below on "the borrowing region".) However, should the investor get near bankruptcy, i.e., $x \downarrow 0$, with little time remaining, i.e., $t \uparrow T$, then the investor must take a fairly active position in the stocks since then the denominator is also going to O.

Let $z(x,t;T,b)$ be defined by

$$z(x,t;T,b) := \frac{x}{be^{-\int_t^T r(s)ds}}, \quad \text{for} \quad x \leq be^{-\int_t^T r(s)ds} \tag{5.1}$$

whereby $0 \leq z \leq 1$, and the interpretation of z is the *percentage of the goal reached by time t*. (Note that this is not the percentage of b reached, but rather the *time-dependent* goal $be^{-\int_t^T r(s)ds}$.) Then we may write the optimal policy as

$$\boldsymbol{f}_t^*(x;b) = \left[\frac{\boldsymbol{\sigma}^{-1}(t)'\boldsymbol{\theta}(t)}{\sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)ds}} \right] \left[\frac{\phi\left(\Phi^{-1}(z)\right)}{z} \right] x.$$

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Introducing now the variable

$$\nu := \Phi^{-1}(z)$$

allows us to rewrite this in terms of v as

$$f_t^*(x;b) = \left[\frac{\sigma^{-1}(t)'\theta(t)}{\sqrt{\int_t^T \theta(s)'\theta(s)ds}}\right] \left[\frac{\phi(\nu)}{\Phi(\nu)}\right] x. \tag{5.2}$$

Note that ν is a monotonically increasing function of z, and that as z goes from 0 to 1, ν goes from $-\infty$ to ∞. As wealth, x, gets closer to the (effective) goal, $be^{-\int_t^T r(s)ds}$, $z \uparrow 1$ and hence $\nu \uparrow +\infty$, while as wealth gets closer to the bankruptcy barrier, 0, we have $z \downarrow 0$ and accordingly $\nu \downarrow -\infty$. Notice too that the function $\phi(\nu)/\Phi(\nu)$ is positive and decreasing in ν.

Equation (5.2) represents the optimal investment strategy as a linear function of wealth (x) where the linear multiple decomposes into a product of two distinct components: the first component is the vector of purely time-dependent elements, $\sigma^{-1}(t)'\theta(t)/\sqrt{\int_t^T \theta(s)'\theta(s)ds}$; and the second is a scalar determined solely by the percentage of the (effective) goal currently achieved, $\phi(\nu)/\Phi(\nu)$. It is easy to see that for a fixed t, as wealth gets closer to the effective goal, and so z increases as does v, this latter quantity decreases. The effect of increasing t on the former quantity is not as clear to check since we have allowed all parameters to be time dependent. For the special case of *constant coefficients*, this time-dependent vector is simply $\sigma^{-1'}\theta/\sqrt{\theta'\theta(T-t)}$, which is clearly increasing as $t \uparrow T$. We will examine the tradeoff between these two effects directly.

5.1 The borrowing region

It is interesting to examine the nature of risk-taking dictated by the policy described above. One manifestation of risk-taking is in the amount of borrowing required by a policy. Here, we examine this dimension of the behavior required by $\{f_t^*\}$ of Theorem 3.1.

Notice first that borrowing takes place only in the region $\Gamma(x,t) := \{x : \mathbf{1}'f_t^*(x;b) \geq x\}$, since $\mathbf{1}'f_t^*(x;b)$ is the total amount of money invested in the risky stocks. This region can be characterized more explicitly by using (5.2). In particular, (5.2) shows that the borrowing region, $\Gamma(x,t)$ is equivalent to the region

$$\Gamma := \left\{\nu : \frac{\phi(\nu)}{\Phi(\nu)} \geq q(t,T)\right\},$$

where the (scalar) function q is defined by $q(t,T) := \sqrt{\int_t^T \theta(s)'\theta(s)\,ds}/\mathbf{1}'\sigma^{-1}(t)'\theta(t)$, and where the variable $\nu = \nu(x,t)$ is defined by $\nu := \Phi^{-1}(z)$.

For the case of *constant coefficients*, $q(t,T)$ reduces to $q(T-t) := \xi \cdot \sqrt{(T-t)}$ where the constant

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ξ is defined by

$$\xi := \frac{\sqrt{(\mu - r\mathbf{1})' \Sigma^{-1} (\mu - r\mathbf{1})}}{\mathbf{1}' \Sigma^{-1} (\mu - r\mathbf{1})} \equiv \frac{\sqrt{\pi^{*\prime} \Sigma \pi^*}}{\mathbf{1}' \pi^*},$$

which in the single-stock case reduces to $\xi = \sigma$.

If we set $\sqrt{\tau} := q(T - t) \equiv \xi \sqrt{T - t}$, then we can consider τ to be the *risk adjusted time to play*, since it is in fact just the time to play $(T - t)$ multiplied by a risk factor (ξ^2). The borrowing region is then equivalent to

$$\Gamma(\nu, \tau) := \{\nu : \phi(\nu)/\Phi(\nu) \geq \sqrt{\tau}\}.$$

To analyze the boundary of this region, let $\nu^*(\tau)$ denote the root to the equation $\phi(\nu)/\Phi(\nu) - \sqrt{\tau} = 0$, i.e., $\nu^*(\tau)$ is the unique number such that

$$\phi(\nu^*)/\Phi(\nu^*) = \sqrt{\tau}.$$

(Since the LHS is decreasing in ν while the RHS is increasing in τ, it is clear that there is a unique root.) Furthermore, it is easy to establish that these roots are *decreasing* in the remaining time, τ, i.e., $\nu^*(\tau) > \nu^*(\tau + \delta)$, for all $\delta > 0$.

The borrowing region is thus as follows: borrowing occurs when $\nu(\tau) < \nu^*(\tau)$, but not if $\nu(\tau) > \nu^*(\tau)$. Equivalently, since $\Phi(\cdot)$ is an increasing invertible function, we see that borrowing occurs only when $\Phi(\nu(\tau)) < \Phi(\nu^*(\tau))$, i.e., when $z(\tau) < z^*(\tau)$, where $z(\tau)$ has the interpretation of being the *proportion of the goal obtained with τ* (risk-adjusted) *time units left to go*. A graph of $z^*(\tau)$ (calculated and drawn in MAPLE V) is given below in Figure 1.
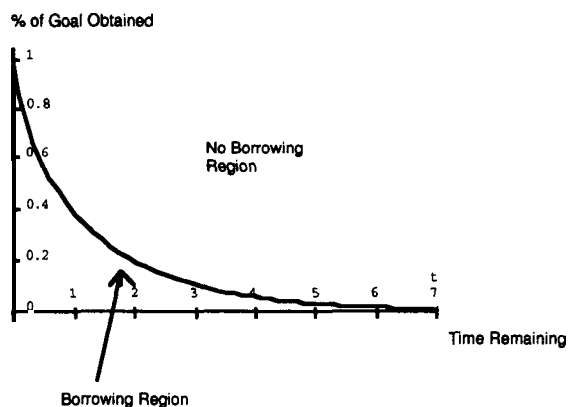

Figure 1. $z^*(\tau)$ plotted against τ.

Some select values of $\tau, \nu^*(\tau)$ and $z^*(\tau)$ are given in Table 1. As Table 1 shows, if there is .05 time units left until the deadline, the investor must borrow under policy f^* unless the investor's

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τ	$\nu^*(\tau)$	$z^*(\tau)$	τ	$\nu^*(\tau)$	$z^*(\tau)$
.001	2.256996187	.9879958435			
.05	1.186694106	.8823258496	.55	.08978236088	.5357699172
.10	.9249416566	.8225018716	.60	.03681409382	.5146833818
.15	.7547039235	.7747866741	.65	-.01307315143	.4947847157
.20	.6247149806	.7339209304	.70	-.06029250311	.4759613364
.25	.5179127160	.6977404152	.75	-.1051762297	.4581179860
.30	.4263687842	.6650804281	.80	-.1479958531	.4411730205
.35	.3457091853	.6352193598	.85	-.1889763275	.4250556853
.40	.2732480356	.6076687214	.90	-.2283062760	.4097040730
.45	.2072097020	.5820769547	.95	-.2661455387	.3950635622
.50	.1463549786	.5581794168	1.00	-.3026308407	.3810856042
1.5	-.6129889502	.2699418230	3.5	-1.419901384	.0778181965
2.0	-.8600276797	.1948868923	4.0	-1.571857688	.0579917796
2.5	-1.069577667	.1424047261	4.5	-1.712651966	.0433882934
3.0	-1.253906116	.1049380659	5.0	-1.844316203	.0325685358

Table 1: Borrowing Region: τ=(risk-adjusted) time to go, $z^*(\tau)$=% of distance to goal by τ. $\nu^*(\tau)$ is the root to the equation $\phi(\nu)/\Phi(\nu) - \tau^{1/2} = 0$.

wealth is already 88.23% of the distance to the goal. As the time to go increases, the investor needs to borrow only at lower percentages. For example, if there is $\tau = 1$ unit of time left to go, then the investor will borrow only if his wealth at that time is less than 38.1% of the way there.

It is important to note that *increasing the risk factor, ξ^2, has the same effect as increasing the actual time left to play, $T - t$*. Therefore, for a higher risk factor, one would borrow less, in the hopes of reaching the goal later.

Remark 5.1: Asymptotics, near the barriers. When wealth, x, is close to the barriers, 0 or $be^{-\int_t^T r(s)ds}$, then z is correspondingly near 0 or 1, and ν is correspondingly near $-\infty$ or $+\infty$. It is interesting to examine what happens to the state-dependent factor, $\phi(\nu)/\Phi(\nu)$, near these boundaries. Note that since ϕ is symmetric, the behavior of $\phi(\nu)/\Phi(\nu)$ as $\nu \downarrow -\infty$ is equivalent to the behavior of $\phi(\nu)/(1 - \Phi(\nu))$ as $\nu \uparrow +\infty$.

Therefore the following two asymptotic results are immediate:

$$\lim_{z \to 1} \frac{\phi(\nu)}{\Phi(\nu)} = \lim_{\nu \uparrow +\infty} \frac{\phi(\nu)}{\Phi(\nu)} = 0$$

$$\lim_{z \to 0} \frac{\phi(\nu)}{\Phi(\nu)} = \lim_{\nu \downarrow -\infty} \frac{\phi(\nu)}{\Phi(\nu)} = \lim_{\nu \uparrow +\infty} \frac{\phi(\nu)}{1 - \Phi(\nu)} = +\infty .$$

As expected, we see that as wealth approaches the goal, the state dependent factor goes to O, but as wealth approaches O, the state dependent factor increases without bound even though,

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as we observed previously, total investment in the risky stocks actually decreases to O in this case.

5.2 **Comparison with Utility Maximizing Policies**

It is interesting to compare this behavior with that of an investor whose objective is to maximize terminal utility from wealth. For example, consider the case where the investor wants to maximize $E\left[u(X_T^f)\right]$, where

$$u(x) = \frac{\delta}{\delta - 1} x^{1 - \frac{1}{\delta}}, \quad \text{for } x > 0, \delta > 0.$$

(Note that this includes logarithmic utility, when = 1.) This power utility function has constant relative risk aversion $\frac{1}{\delta}$. The optimal policy for this case, call it $\{f_t^\delta, 0 \leq t \leq T\}$, is the vector (cf. Merton [20], Karatzas [14])

$$f_t^\delta(x) = \delta \cdot \Sigma^{-1}(t) \left(\mu(t) - r(t)\mathbf{1}\right) \cdot x \equiv \delta \pi_t^* \cdot x,$$

wher π_t^* is the optimal growth policy discussed earlier. The utility maximizing investor invests more heavily in the risky stocks, relative to the probability-maximizing investor, when $f_t^\delta(x) > f_t^*(x; b)$, and vica versa. It is easily seen that this occurs for values (x, t) for which

$$\frac{\phi(\nu)}{\Phi(\nu)} \leq \delta \sqrt{\int_t^T \theta(s)' \theta(s) ds}$$

and vica versa. Thus the dynamics of this comparison reduce essentially to that described above by the borrowing region, modified by the risk aversion parameter δ.

6 Proof of Theorem 3.1

In this section, we provide the proof of Theorem 3.1. We will first show that the function V satisfies the appropriate Hamilton-Jacobi-Bellman (HJB) equations of stochastic control theory and then employ a martingale argument to verify optimality. This will prove the Theorem as well as Corollary 3.2. We then show how we obtained the candidate value function by extending the elegant argument of Heath [11] to our case.

6.1 Verification of Optimality

Standard arguments in control theory (see e.g., Fleming and Rishel [8, Example 2, p. 161]) shows that the appropriate Hamilton-Jacobi-Bellman (HJB) optimality equation for *V* is

$$\sup_f \left\{ \mathcal{A}^f V(t, x) \right\} = 0 \tag{6.1}$$

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subject to the boundary conditions

$$
V(t,x;b) = \begin{cases} 1 & \text{for } x \geq be^{-\int_t^T r(s)ds}, t \leq T \\ 0 & \text{for } x = 0, t \leq T \\ I_{\{b\}} & \text{for } t = T. \end{cases} \tag{6.2}
$$

The generator of (2.5) shows that the HJB optimality equation (6.1) is

$$
\sup_{\boldsymbol{f}} \left\{ V_t + (\boldsymbol{f}_t'(\boldsymbol{\mu}(t) - r(t)\mathbf{1}) + r(t)x)V_x + \frac{1}{2}\boldsymbol{f}_t'\boldsymbol{\Sigma}(t)\boldsymbol{f}_t V_{xx} \right\} = 0. \tag{6.3}
$$

Assuming that a classical solution to (6.3), say V, exists and satisfies $V_x > 0$, $V_{xx} < 0$ for $x < be^{-\int_t^T r(s)ds}$, we may then optimize with respect to $\boldsymbol{f}_t$ in (6.3) to obtain the maximizer

$$
\boldsymbol{f}_t^*(x;b) = -\boldsymbol{\Sigma}^{-1}(t)(\boldsymbol{\mu}(t) - r(t)\mathbf{1})\frac{V_x}{V_{xx}} \equiv -\boldsymbol{\sigma}^{-1}(t)'\boldsymbol{\theta}(t)\frac{V_x}{V_{xx}}. \tag{6.4}
$$

When (6.4) is then placed back into (6.3) and the resulting equation simplified, we find that (6.3) is equivalent to the nonlinear partial differential equation

$$
V_t + r(t)xV_x - \frac{1}{2}\boldsymbol{\theta}(t)'\boldsymbol{\theta}(t)\frac{V_x^2}{V_{xx}} = 0, \quad \text{for } t < T, \quad \text{and } x < be^{-\int_t^T r(s)ds} \tag{6.5}
$$

subject to the (discontinuous) boundary condition (6.2).

Recalling now the basic facts about the normal pdf and cdf:

$$
\frac{d\Phi(y)}{dy} = \phi(y); \quad \frac{d\Phi^{-1}(y)}{dy} = \frac{1}{\phi(\Phi^{-1}(y))}; \quad \frac{d\phi(y)}{du} = -y\phi(y); \tag{6.6}
$$

it can be verified that for the function $V(t,x;b)$ of (3.1), we have

$$
V_x = \phi\left(\Phi^{-1}\left(\frac{x}{b}e^{\int_t^T r(s)ds}\right) + \sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,ds}\right)\frac{1}{b}e^{\int_t^T r(s)ds}\left[\phi\left(\Phi^{-1}\left(\frac{x}{b}e^{\int_t^T r(s)ds}\right)\right)\right]^{-1} \tag{6.7}
$$

$$
V_{xx} = -V_x\left[\sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)ds}\,\frac{1}{b}e^{\int_t^T r(s)ds}\left[\phi\left(\Phi^{-1}\left(\frac{x}{b}e^{\int_t^T r(s)ds}\right)\right)\right]^{-1}\right] \tag{6.8}
$$

$$
V_t = -V_x\left[r(t)x + \frac{1}{2}\frac{\boldsymbol{\theta}(t)'\boldsymbol{\theta}(t)}{\sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)ds}}\phi\left(\Phi^{-1}\left(\frac{x}{b}e^{\int_t^T r(s)ds}\right)\right)be^{-\int_t^T r(s)ds}\right]. \tag{6.9}
$$

When these derivatives are placed back into (6.5), it is seen that in fact V solves (6.5), and so the optimal control $\boldsymbol{f}_t^*(x)$ of (3.2) is obtained by placing (6.7) and (6.8) into (6.4).

Moreover, it is also seen that the first two boundary conditions in (6.2) are satisfied for V of (3.1). This follows from the fact that Φ is a cdf, and hence $\Phi^{-1}(u) = -\infty$ for $u \leq 0$, $\Phi^{-1}(u) = \infty$ for $u \geq 1$. The third boundary condition of (6.2) causes a *discontinuity*. This problem (for

"probability maximizing objectives") is discussed in Fleming and Rishel [8, Example 2, Page 161]; where it is shown that such a discontinuity y is acceptable, provided that the optimal wealth process, $\{X_t^*, t \leq T\}$ (i.e., the wealth under the control f_t^*), satisfies the condition $P(0 < X_T^* < b) = 0$. (That is, the optimal terminal wealth must be equal to one of the two barriers, 0 or b.) We will show directly and rigorously that this condition is in fact met here, although it is easy to see that the only way we could have $\int_0^T f_t^{*'} f_t^* dt < \infty$ is if in fact $X_T^* = 0$ or $X_T^* = b$.

To see this more clearly, let $H_t := z(X_t^*, t; T, b)$, where the function z was defined earlier in (5.1), and where X_t^* denotes the optimal wealth process; i.e., $H_t = X_t^* / \left[be^{-\int_t^t r(s)ds} \right]$, and X_t^* is determined by the stochastic differential equation in (3.6). An application of Ito's formula to (3.6) then shows that

$$dH_t = \frac{\theta(t)'\theta(t)}{\sqrt{\int_t^T \theta(s)'\theta(s)ds}} \phi\left(\Phi^{-1}(H_t)\right) dt + \frac{1}{\sqrt{\int_t^T \theta(s)'\theta(s)ds}} \phi\left(\Phi^{-1}(H_t)\right) \theta(t)'dW_t. \qquad (6.10)$$

Another application of Ito's formula will verify that the solution to (6.10) is

$$H_t = \Phi\left(\frac{\int_0^t \theta(s)'dW_s + \int_0^t \theta(s)'\theta(s)ds + \sqrt{\int_0^T \theta(s)'\theta(s)ds}\, \Phi^{-1}(H_0)}{\sqrt{\int_t^T \theta(s)'\theta(s)ds}} \right), \qquad (6.11)$$

from which the representation of X_t^* given in (3.7) is immediate, proving Corollary 3.2.

Inspection of (6.11) shows that H_T is either O or 1, and that moreover,

$$P(H_T = 1) = P\left(\int_0^T \theta(s)'dW_s \geq -\int_0^T \theta(s)'\theta(s)ds - \sqrt{\int_0^T \theta(s)'\theta(s)ds}\, \Phi^{-1}(H_0) \right)$$

$$\equiv \Phi\left(\Phi^{-1}(H_0) + \sqrt{\int_0^T \theta(s)'\theta(s)ds} \right).$$

Substituting now for X_T^* shows that

$$P(X_T^* = b) = V(0, X_0; b) = 1 - P(X_T^* = 0)$$

which is the desired conclusion.

Note further that (6.10) can be rewritten using $\tilde{W}_t$ as

$$dH_t = \frac{\phi\left(\Phi^{-1}(H_t)\right)}{\sqrt{\int_t^T \theta(s)'\theta(s)ds}} \theta(t)'d\tilde{W}_t \equiv \left[be^{-\int_0^t r(s)ds} \right]^{-1} f_t^{*'} \sigma(t) d\tilde{W}_t \qquad (6.12)$$

which shows that H_t is a $\tilde{P}$ martingale (recall that $0 \leq H_t \leq 1$). As such, the martingale representation theorem (e.g., [16, Theorem 3.4.2]) provides the representation $H_t = H_0 + \int_0^t \xi_s' d\tilde{W}_s$ with ξ a measurable, adapted process that is square-integrable, i.e., $\int_0^T \xi_s' \xi_s ds < \infty$ a.s. We may therefore conclude immediately that $\int_0^T f_t^{*'} f_t^* dt < \infty$. ∎

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6.2 Obtaining the candidate value function

While we have already shown that V does indeed solve the HJB equation, we have not actually solved the resulting nonlinear partial differential equation (6.5) directly for V. Rather, we obtained a candidate solution to (6.5) by reducing the problem to a form whose optimal value function we were able to guess by extending an argument of Heath [11], as we now show.

To proceed, set $Y_t^f \equiv e^{-\int_0^t r(s)ds} X_t^f$, and then apply Ito's formula to get

$$dY_t^f = e^{-\int_0^t r(s)ds} \left[\boldsymbol{f}_t'(\boldsymbol{\mu}(t) - r(t)\mathbf{1})dt + \sum_{i=1}^n \sum_{j=1}^n f_t^{(i)} \sigma_{ij}(t)dW_t^{(j)} \right] \qquad (6.13)$$

Setting now $\boldsymbol{g}_t \equiv \boldsymbol{f}_t e^{-\int_0^t r(s)ds}$, let us first consider the following problem: *Find $U(t,y;c) := \sup_{\boldsymbol{g}} P_{(t,y)}(Y_T^g \geq c)$ and its associated optimal control vector, $\{\boldsymbol{g}_t^*, t \leq T\}$, where*

$$dY_t^g = \boldsymbol{g}_t'(\boldsymbol{\mu}(t) - r(t)\mathbf{1})dt + \sum_{i=1}^n \sum_{j=1}^n g_t^{(i)} \sigma_{ij}(t)dW_t^{(j)} . \qquad (6.14)$$

This is now essentially (aside from scaling factors) the multivariate generalization of the problem studied in Kulldorff [19] and Heath [11]. Following Heath [11], we note that we can rewrite (6.14) in terms of the $\tilde{P}$ Brownian motion $\tilde{\boldsymbol{W}}_t = \boldsymbol{W}_t + \int_0^t \boldsymbol{\theta}(t)dt$, as

$$dY_t^g = \sum_{i=1}^n \sum_{j=1}^n g_t^{(i)} \sigma_{ij}(t)d\tilde{W}_t^{(j)} . \qquad (6.15)$$

It is clear, by the admissibility assumption on $\boldsymbol{f}_t$ — and hence on $\boldsymbol{g}_t$ — that the local martingale Y_t^g is in fact a *martingale* under the measure $\tilde{P}$. Hence, letting $A^g := \{\omega : Y_T^g \geq c\}$, it is clear by the martingale inequality that we have $P(A^g) \leq Y_0/c$, and since the RHS of this is independent of the policy g, we may conclude that $\sup\{\boldsymbol{g} : P(A^g)\} \leq Y_0/c$. If we assume this upper bound can be hit, then a straightforward generalization of an argument in Heath [11], reproduced here for the sake of completeness in the Appendix, then shows that

$$U(t,y;c) = \Phi\left(\Phi^{-1}\left(\frac{y}{c}\right) + \sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)ds} \right) . \qquad (6.16)$$

A HJB argument then shows that $\boldsymbol{g}_t^* = -\boldsymbol{\Sigma}^{-1}(t)(\boldsymbol{\mu}(t) - r(t)\mathbf{1})\frac{U_y}{U_{yy}}$, and by differentiating (6.16) appropriately we obtain

$$\boldsymbol{g}_t^*(y;c) = \left[\frac{\boldsymbol{\Sigma}^{-1}(t)(\boldsymbol{\mu}(t) - r(t)\mathbf{1})}{\sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)ds}} \right] c \cdot \phi\left(\Phi^{-1}\left(\frac{y}{c}\right) \right) . \qquad (6.17)$$

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All we need do now is recognize that

$$\sup_{\boldsymbol{f}} P_{(t,x)}\left(X_T^f \geq b\right) \equiv \sup_{\boldsymbol{g}} P_{(t,y)}\left(Y_T^g \geq e^{-\int_0^T r(s)ds}b\right) \tag{6.18}$$

with

$$y = xe^{-\int_0^t r(s)ds}, \quad \text{and} \quad \boldsymbol{f}_t^* \equiv e^{\int_0^t r(s)ds}\boldsymbol{g}_t^*.$$

And indeed taking $y = e^{-\int_0^t r(s)ds}x$ and $c = be^{\int_0^T r(s)ds}$ in (6.16) and (6.17) gives

$$V(t,x;b) = U\left(t, xe^{-\int_0^t r(s)ds}; be^{-\int_0^T r(s)ds}\right)$$

where $U(t,\bullet;\bullet)$ is the function given by (6.16), and

$$\boldsymbol{f}_t^*(x;b) = e^{\int_0^t r(s)ds}\boldsymbol{g}_t^*(xe^{-\int_0^t r(s)ds}; be^{-\int_0^T r(s)ds}),$$

where $\boldsymbol{g}_t^*(\bullet;\bullet)$ is defined by (6.17).

Of course, this argument still needs a rigorous verification, which is the content of the previous proof. The functions U and $\boldsymbol{g}_t^*$ in (6.16) and (6.17) are central to the remainder of the paper.

7 Exogenous Income

In this section we analyze the case where the investor has an exogenous (deterministic) income stream. Specifically, suppose that income is earned at rate $\{l(t), 0 \leq t \leq T\}$, where $l(s) \geq 0$ for all s. The investor's wealth then evolves according to

$$\begin{aligned}
dX_t^f &= \sum_{i=1}^n f_t^{(i)}\frac{dS_i(t)}{S_i(t)} + \left(X_t^f - \sum_{i=1}^n f_t^{(i)}\right)\frac{dB_t}{B_t} + l(t)\,dt \\
&= \left[rX_t + \sum_{i=1}^n f_t^{(i)}(\mu_i(t) - r(t)) + l(t)\right]dt + \sum_{i=1}^n\sum_{j=1}^n f_t^{(i)}\sigma_{ij}(t)dW_t^{(j)}. \tag{7.1}
\end{aligned}$$

This case can be treated by modifying our previous analysis, as we now show. To that end, define first the function $\kappa(x,t)$ and the constant Λ by

$$\kappa(x,t) := e^{-\int_0^t r(u)du}x - \int_0^t e^{-\int_0^s r(u)du}l(s)ds. \tag{7.2}$$

$$\Lambda := be^{-\int_0^T r(u)du} - \int_0^T e^{-\int_0^s r(u)du}l(s)ds. \tag{7.3}$$

The optimal value function and optimal policy for maximizing the probability that the investor achieves the wealth level b by the terminal time T are given in the following theorem.

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Theorem 7.1 *For an* investor *whose wealth process follows (7.1),*

$$\Psi(t,x;b,T) := \sup_{f} P_{(t,x)}\left(X_T^f \geq b\right) = \Phi\left(\Phi^{-1}\left(\frac{\kappa(x,t)}{\Lambda}\right) + \sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)ds}\right) \qquad (7.4)$$

and the associated optimal investment policy is

$$\boldsymbol{f}_t^{(l)}(x;b) := \arg\sup_{f} P_{(t,x)}\left(X_T^f \geq b\right) = \left[\frac{\boldsymbol{\Sigma}^{-1}(t)(\boldsymbol{\mu}(t) - r(t)\mathbf{1})}{\sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)ds}}\right]\left[e^{\int_0^t r(u)du}\Lambda\right]\phi\left(\Phi^{-1}\left(\frac{\kappa(x,t)}{\Lambda}\right)\right).$$

$$(7.5)$$

Remark 7.1: An analysis similar to that done after Theorem 3.1 shows that under this policy, the investor stops investing in the risky stocks altogether, i.e., $\boldsymbol{f}_t^{(l)}(x;b) = \mathbf{0}$, when $\kappa(x,t) \geq \Lambda$ as well as when $\kappa(x,t) \leq 0$. The former case is in effect for wealth levels, x, that satisfy

$$x \geq be^{-\int_t^T r(u)du} - \int_t^T e^{\int_t^s r(u)du}l(s)ds, \qquad (7.6)$$

while the latter case occurs for wealth levels that satisfy

$$x \leq \int_0^t e^{\int_s^t r(u)du}l(s)ds. \qquad (7.7)$$

The economic interpretation of (7.6) is clear since we may rewrite it as

$$xe^{\int_t^T r(u)du} + \int_t^T e^{\int_s^T r(u)du}l(s)ds \geq b$$

which shows directly that by investing current wealth (x) and all future income in the riskless asset, the goal b *can* be reached with certainty at time T.

The interpretation of (7.7) is perhaps more interesting, since the RHS of (7.7) is the level of wealth that would have been achieved at time t by simply investing all the income until that time directly into the risk-free asset. Thus if wealth ever falls to that level, the investor following policy $\boldsymbol{f}_t^{(l)}$ would stop investing in the risky stocks and just put all wealth into the risk-free asset. Therefore, as opposed to the case without income treated earlier, wealth is effectively bounded away from 0, and so the investor with positive income can never go bankrupt under policy $\boldsymbol{f}_t^{(l)}$. Moreover, this lower bound on wealth acts as a sort of "performance bound" on the investor: if the performance of the risky stocks is such that wealth drops to the value of the income stream invested in the risk-free asset, investment in the risky stocks stop.

This behavior differs significantly from the behavior of a utility maximizer with an income stream. For example, in the constant coefficient case with $l(s) = l$, it can be shown that

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the optimal policy for an investor who wishes to maximize expected logarithmic utility of terminal wealth at time 2', is to invest

$$\Sigma^{-1} \left(\mu - r\mathbf{1}\right) \left(x + \frac{l}{r}\left(1 - e^{-r(T-t)}\right)\right)$$

in the risky stocks (cf. Merton [20]). Thus a utility maximizer essentially capitalizes future earnings (till T) at the risk-free rate and then invests as if his current fortune already included this amount. Therefore, even if wealth goes *negative*, so long as $x > -\frac{l}{r}\left(1 - e^{-r(T-t)}\right)$, the investor still invests a positive amount in the risky stocks. Not so for the probability maximizer who behaves in the manner described above. In essence, a utility maximizer uses income to take extra risks, by borrowing against future earnings, while our Theorem 7.1 shows that a probability maximizer relies on income to become more cautious.

Proof: Let $g_t := e^{-\int_0^t r(s)ds} f_t$, and for the remainder of this section only, let Y_t^g be defined now by

$$Y_t^g := e^{-\int_0^t r(u)du} X_t^f - \int_0^t e^{-\int_0^s r(u)du} l(s)ds \qquad (7.8)$$

where X_t^f is as in (7.1). Then a simple application of Ito's formula shows that

$$dY_t^g = g_t'(\mu(t) - r(t)\mathbf{1})dt + \sum_{i=1}^{n}\sum_{j=1}^{n} g_t^{(i)}\sigma_{ij}(t)dW_t^{(j)} \equiv \sum_{i=1}^{n}\sum_{j=1}^{n} g_t^{(i)}\sigma_{ij}(t)d\tilde{W}_t^{(j)} \qquad (7.9)$$

which is the same as (6.13) and (6.14). Thus we observe immediately that if

$$U(t, y; c) = \sup_{g} P\left(Y_T^g \geq c | Y_t = y\right),$$

with optimizer g_t^*, then $U(t, y; c)$ and g_t^* are given by (6.16) and (6.17). Now, by (7.8), we see that

$$\sup_{g} P\left(Y_T^g \geq c | Y_t = y\right) \equiv \sup_{f} P\left(X_T^f \geq ce^{\int_0^T r(u)du} + \int_0^T e^{\int_s^T r(u)du} l(s)ds \mid Y_t = \kappa(x, t)\right) \qquad (7.10)$$

where $\kappa(x, t)$ is given by (7.2). Thus we may invert this to deduce that in terms of a goal problem for X_t^f given by (7.1), with arbitrary b, we have

$$\sup_{f} P\left(X_T^f \geq b \mid X_t = x\right) = U\left(t, \kappa(x, t); \Lambda\right),$$

as well as

$$f_t^{(l)}(x; b) = e^{\int_0^t r(u)du} g_t^*\left(\kappa(x, t); \Lambda\right),$$

where the function $U(t, \bullet; \bullet)$ is defined by (6.16), $g_t^*(\bullet; \bullet)$ is defined by (6.17), $\kappa(x, t)$ by (7.2) and Λ by (7.3). Direct substitution then shows that $U(t, \kappa(x, t); \Lambda) = \Psi(t, x; b, T)$ of (7.4) and that the policy given in (7.5) is indeed optimal for the problem of maximizing the probability of attaining the goal b by time T. ∎

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8 Liabilities

Consider now an investor who faces a deterministic liability stream $\{L(t), 0 \le t \le T\}$, whereby an amount, $L(s) \ge 0$, must be *paid out* at time s. This model is applicable to scenarios such as pension fund management. The reserves of the pension fund evolve according to

$$
\begin{aligned}
dX_t^f &= \sum_{i=1}^n f_t^{(i)} \frac{dS_i(t)}{S_i(t)} + \left(X_t^f - \sum_{i=1}^n f_t^{(i)}\right)\frac{dB_t}{B_t} - L(t)\,dt \\
&= \left[rX_t + \sum_{i=1}^n f_t^{(i)}(\mu_i(t) - r(t)) - L(t)\right]dt + \sum_{i=1}^n \sum_{j=1}^n f_t^{(i)}\sigma_{ij}(t)dW_t^{(j)}.
\end{aligned}
\tag{8.1}
$$

A variety of investment problems related to the achievement of goals for this model in an infinite horizon setting was studied in Browne [4] for the constant coefficients case.

It is tempting to merely mimic our previous analysis for this case, however, we would then get a policy under which wealth can become negative. To see this, let $g_t := e^{-\int_0^t r(s)ds}f_t$, and for the remainder of this section only, let Y_t^g be defined now by

$$
Y_t^g := e^{-\int_0^t r(u)du}X_t^f + \int_0^t e^{-\int_0^s r(u)du}L(s)ds
\tag{8.2}
$$

where X_t^f is as in (8.1). Then once again it is clear that

$$
dY_t^g = g_t'(\mu(t) - r(t)\mathbf{1})dt + \sum_{i=1}^n \sum_{j=1}^n g_t^{(i)}\sigma_{ij}(t)dW_t^{(j)} \equiv \sum_{i=1}^n \sum_{j=1}^n g_t^{(i)}\sigma_{ij}(t)d\tilde{W}_t^{(j)}
\tag{8.3}
$$

which again is the same as (6.13) and (6.14). Thus we observe immediately that if

$$
U(t, y; c) = \sup_{g} P\left(Y_T^g \ge c | Y_t = y\right),
$$

with optimizer g_t^*, then $U(t, y; c)$ and g_t^* are given by (6.16) and (6.17). Now, by (8.2), we see that

$$
\sup_{g} P\left(Y_T^g \ge c | Y_t = y\right) \equiv \sup_{f} P\left(X_T^f \ge c e^{\int_0^T r(u)du} - \int_0^T e^{\int_s^T r(u)du}L(s)ds \mid Y_t = \beta(x,t)\right)
\tag{8.4}
$$

where now

$$
\beta(x, t) := e^{-\int_0^t r(u)du}x + \int_0^t e^{-\int_0^s r(u)du}L(s)ds.
\tag{8.5}
$$

We may again invert this to deduce that in terms of a goal problem for X_t^f – now determined by (8.1) - with arbitrary b,

$$
\sup_{f} P\left(X_T^f \ge b \mid X_t = x\right) = U\left(t, \beta(x,t); \eta\right),
\tag{8.6}
$$

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where the function $U(t, \bullet; \bullet)$ is given by (6.16), $\beta(x, t)$ is defined above and

$$\eta := be^{-\int_0^T r(u)du} + \int_0^T e^{-\int_0^s r(u)du} L(s)ds.$$

(8.7)

We may therefore deduce directly that the policy given by by

$$f_t^{(L)}(x; b) = e^{\int_0^t r(u)du} g_t^* \left(\beta(x, t); \eta\right)$$

is indeed optimal for the problem of maximizing *the probability of attaining the* goal *b by time T*. However, there is a new problem for this model which was not in effect in the cases treated earlier, namely, under policy $f^{(L)}$, it is *possible for wealth to become negative*. In fact, under $f^{(L)}$, the investor will still invest so long as $\beta(x, t) > 0$, which in terms of wealth means as long as

$$x > -\int_0^t e^{\int_s^t r(u)du} L(s)ds.$$

(8.8)

This of course causes no problems in terms of the general control problem that we consider, since it simply means that under policy $f_t^{(L)}$, the investor, or pension fund manager, needs to *borrow* from the bank whenever wealth is negative so long as (8.8) is met. However, suppose there is also a hard nonnegativity constraint on wealth, what should the pension fund manager do ? While we don't have a complete answer to this problem yet, we can address the problem of how the pension fund manager should invest subject to the regulatory constraint that *current reserves*, X_t^f, *always be greater than the current discounted value of future liabilities*, i.e.,

$$X_t^f \geq \int_t^T e^{-\int_t^s r(u)du} L(s)ds.$$

(8.9)

We give the answer in the following theorem. To ease notation somewhat, let

$$\tilde{L}(t) := \int_t^T e^{-\int_t^s r(u)du} L(s)ds, \quad \text{and let} \quad \tilde{b}(t) := be^{-\int_t^T r(u)du}.$$

(8.10)

Theorem 8.1 *Consider an investor whose wealth, X_t^f, is given by (8.1), and whose objective is to maximize $P(X_T^f \geq b)$, subject to the regulatory constraint in (8.9), i.e., $X_t^f \geq \tilde{L}(t)$ for all $t \leq T$.*

Then, provided that initial wealth, X_0, is such that $X_0 \geq \tilde{L}(0)$, the optimal value function as given *by*

$$G(t, x; b) := \sup_f P_{(t,x)}\left(X_T^f \geq b, \inf_{t \leq u \leq T} X_u^f \geq \tilde{L}(u)\right) = \Phi\left(\Phi^{-1}\left(\frac{x - \tilde{L}(t)}{\tilde{b}(t)}\right) + \sqrt{\int_t^T \theta(s)'\theta(s)ds}\right)$$

(8.11)

and *the associated optimal investment policy is given by*

$$\tilde{f}_t(x; b) = \left[\frac{\Sigma^{-1}(t)(\mu(t) - r(t)\mathbf{1})}{\sqrt{\int_t^T \theta(s)'\theta(s)ds}}\right] \tilde{b}(t)\phi\left(\Phi^{-1}\left(\frac{x - \tilde{L}(t)}{\tilde{b}(t)}\right)\right),$$

(8.12)

where the functions $\tilde{L}(t)$ and $\tilde{b}(t)$ are defined in (8.10).

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Remark 8.2: Observe the 'portfolio insurance' type of behavior that is implicit in (8.12), in that it is only the *surplus* of wealth over the 'floor', $x - \tilde{L}(t)$, rather than wealth itself that plays a role in the policy.

Proof: Recognize first that (8.2) shows that the requirement that $X_t^f \geq \tilde{L}(t)$, for all $0 \leq t \leq T$ is equivalent to the requirement that Y_t^g be greater than the *constant* $\tilde{L}(0)$, for all $0 \leq t \leq T$, where Y_t^g is defined by (8.2), and $\tilde{L}(0) \equiv \int_0^T e^{-\int_0^s r(u)du} L(s)ds$. For arbitrary c with $L(0) \leq y \leq c$, let

$$\tilde{U}(t, y; c) := \sup_{g} P_{(t,y)} \left(Y_T^g \geq c, \inf_{t \leq s \leq T} Y_s^g \geq \tilde{L}(0) \right),$$

with optimal control function

$$\tilde{g}_t^*(y; c) := \arg\sup_{g} P_{(t,y)} \left(Y_T^g \geq c, \inf_{t \leq s \leq T} Y_s^g \geq \tilde{L}(0) \right).$$

The change of variables $\tilde{Y}_T^g := Y_t^g - \tilde{L}(0)$ shows that

$$\tilde{U}(t, y; c) \equiv U\left(t, y - \tilde{L}(0); c - \tilde{L}(0)\right), \quad \text{and} \quad \tilde{g}_t^*(y; c) \equiv g_t^*\left(y - \tilde{L}(0); c - \tilde{L}(0)\right) \tag{8.13}$$

where $U(t, y; c)$ and $g_t^*(y; c)$ where defined earlier by (6.16) and (6.17).

Observe now that

$$\sup_{f} P_{(t,x)} \left(X_T^f \geq b, \inf_{t \leq u \leq T} X_u^f \geq \tilde{L}(t) \right) \equiv \sup_{g} P_{(t,y)} \left(Y_T^g \geq \eta, \inf_{t \leq u \leq T} Y_u^g \geq \tilde{L}(0) \right)$$

with $y = \beta(x, t)$, where $\beta(x, t)$ is given by (8.5), and η is given by (8.7). As such, (8.13) shows that we must have

$$G(t, x; b) \equiv U\left(t, \beta(x, t) - \tilde{L}(0); \eta - \tilde{L}(0)\right) \tag{8.14}$$

$$\tilde{f}_t(x; b) \equiv e^{\int_0^t r(u)du} g_t^*\left(\beta(x, t) - \tilde{L}(0); \eta - \tilde{L}(0)\right) \tag{8.15}$$

where $U(t, y; c)$ and $g_t^*(y; c)$ are defined by (6.16) and (6.17). Direct substitution of $\beta(x, t)$ from (8.5), η from (8.7) and $\tilde{L}(0)$ from (8.10) into these functions give (8.11) and (8.12). ∎

9 Beating another Portfolio

In this section we apply our earlier results to derive optimal portfolio strategies for an investor, such as a fund manager, who is interested solely in beating a given stochastic *benchmark*. The benchmark is most typically an index, such as the S&P 500. In particular, it is just one specific portfolio strategy. First we show that the problem of beating the actual value of the benchmark

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by a constant absolute amount is just a simple application of our earlier results – but only for the trivial case in which wealth is already greater than the benchmark. Next we consider the more interesting problem of beating the benchmark portfolio by a given percentage, which is the focal point of active portfolio management. We then find the related policy which corrects for the downside risk. When the benchmark portfolio strategy is the *optimal growth policy,* then the ratio of the wealth from any arbitrary portfolio strategy to the benchmark is in fact a *supermartingale,* and as such problems arise. Nevertheless, we are able to find a strategy which does achieve the maximal possible probability of beating the optimal growth policy by a prespecified percentage by a fixed deadline.

9.1 Beating the benchmark by an absolute amount

Suppose now that the investor's wealth, $\left\{ X_t^f, t \le T \right\}$, evolves as in (2.3), and let Q_t denote the value of the benchmark at time t. Then, for some *given strategy,* π_t, we assume that Q_t evolves according to

$$
\begin{aligned}
dQ_t &= \sum_{i=1}^{n} \pi_t^{(i)} \frac{dS_i(t)}{S_i(t)} + \left(Q_t - \sum_{i=1}^{n} \pi_t^{(i)} \right) \frac{dB_t}{B_t} \\
&= \left[rQ_t + \sum_{i=1}^{n} \pi_t^{(i)} (\mu_i(t) - r(t)) \right] dt + \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_t^{(i)} \sigma_{ij}(t) dW_t^{(j)} ,
\end{aligned}
\tag{9.1}
$$

upon substituting from (2.1) and (2.2). Let $\boldsymbol{\pi}_t = (\pi_t^{(1)}, \ldots, \pi_t^{(n)})'$.

If the investor's objective is then to choose a strategy $\boldsymbol{f}$ to *maximize the probability that the benchmark is beaten at time T,* say by the amount κ, i.e.,

$$
\sup_{\boldsymbol{f}} P \left(X_T^f > Q_T + \kappa \, | X_t = x, Q_t = q \right) \equiv \sup_{\boldsymbol{f}} P \left(X_T^f - Q_T > \kappa \, | X_t = x, Q_t = q \right) ,
$$

then, since

$$
d \left(X_t^f - Q_t \right) = \left[r \left(X_t^f - Q_t \right) + \sum_{i=1}^{n} \left(f_t^{(i)} - \pi_t^{(i)} \right) (\mu_i(t) - r(t)) \right] dt + \sum_{i=1}^{n} \sum_{j=1}^{n} \left(f_t^{(i)} - \pi_t^{(i)} \right) \sigma_{ij}(t) dW_t^{(j)}
$$

it is immediate from Theorem 3.1 that, for $0 < X_0 - Q_0 < \kappa$, the optimal policy is

$$
\boldsymbol{f}_t^*(x, q) = \boldsymbol{\pi}_t + \boldsymbol{f}_t^*(x - q; \kappa)
$$

where $\boldsymbol{f}_t^*$ is the policy of (3.2). However, this is applicable *only for $x > q$,* i.e., when the portfolio is already greater than the benchmark.

9.2 Beating the benchmark portfolio by a given percentage

More interesting is the objective of ensuring that X^f exceeds the benchmark Q by a preset percentage. To that end, we will find it more convenient to use a geometric, or *proportional parameterization* of the model, i.e., instead of parameterizing the problem by taking $\boldsymbol{f}_t$ to be the vector of *absolute* amounts invested in the stocks, we instead – for the remainder of this section – let $f_t^{(i)}$ denote the *proportion* of wealth invested in stock i at time t, with $\boldsymbol{f}_t := \left(f_t^{(1)}, \ldots, f_t^{(n)} \right)'$ now denoting the corresponding column vector of proportions. Then under this parameterization, it is clear that the wealth process evolves as (compare with (2.3))

$$
\begin{aligned}
dX_t^f &= X_t^f \sum_{i=1}^{n} f_t^{(i)} \frac{dS_i(t)}{S_i(t)} + X_t^f \left(1 - \sum_{i=1}^{n} f_t^{(i)} \right) \frac{dB_t}{B_t} \\
&= X_t^f \left[r(t) + \sum_{i=1}^{n} f_t^{(i)}(t)(\mu_i(t) - r(t)) \right] dt + X_t^f \sum_{i=1}^{n} \sum_{j=1}^{n} f_t^{(i)}(t) \sigma_{ij}(t) dW_t^{(j)}, \quad (9.2)
\end{aligned}
$$

upon substituting from (2.1) and (2.2).

Similarly let Q_t now be defined by

$$
dQ_t = Q_t \left[r(t) + \sum_{i=1}^{n} \pi_t^{(i)}(\mu_i(t) - r(t)) \right] dt + Q_t \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_t^{(i)} \sigma_{ij}(t) dW_t^{(j)}, \quad (9.3)
$$

where $\boldsymbol{\pi}_t = \left(\pi_t^{(1)}, \ldots, \pi_t^{(n)} \right)'$ is the (column) vector of portfolio weights in the benchmark process.

An interesting objective is then to choose a policy that *maximizes the probability that for a fixed* T, X_T^f *exceeds* Q_t *by a predetermined percentage*. Denote the resulting optimal policy by $\boldsymbol{f}^\pi$, i.e.,

$$
\{\boldsymbol{f}_t^\pi\} = \arg \sup_{\boldsymbol{f}} P \left(X_T^f > (1 + \lambda) \cdot Q_T \right).
$$

Let $Z_t^f(\pi) = X_t^f/Q_t$, with $Z_0^f = X_0/Q_0$. A simple application of Ito's formula then gives

$$
dZ_t^f = Z_t^f (\boldsymbol{f}_t - \boldsymbol{\pi}_t)' \left(\boldsymbol{\mu}(t) - r(t)\boldsymbol{1} - \boldsymbol{\Sigma}(t)\boldsymbol{\pi}_t \right) dt + Z_t^f \sum_{i=1}^{n} \sum_{j=1}^{n} (f_t^{(i)} - \pi_t^{(i)}) \sigma_{ij}(t) dW_t^{(j)}. \quad (9.4)
$$

In the next theorem, we provide the optimal policy and the optimal value function. It depends solely on the current value of the ratio of the two portfolios, Z_t, as it is obvious that

$$
\sup_{\boldsymbol{f}} P \left(X_T^f > (1 + \lambda) Q_T \,|\, X_t^f = x, Q_t = y \right) \equiv \sup_{\boldsymbol{f}} P \left(Z_T^f > 1 + \lambda \,|\, Z_t^f = \frac{x}{y} \right).
$$

Theorem 9.1 *Let* $Z_s^f = X_s^f/Q_s$ *be defined by (9.4). Then*

$$
F(t, z; \lambda, T) := \sup_{\boldsymbol{f}} P \left(Z_T^f > 1 + \lambda \,|\, Z_t = z \right) = \Phi \left(\Phi^{-1} \left(\frac{z}{1 + \lambda} \right) + \sqrt{\int_t^T \gamma(s) ds} \right) \quad (9.5)
$$

and the optimal (proportional) control $\boldsymbol{f}_t^\pi$, defined as $\boldsymbol{f}_t^\pi := \arg\left\{\sup_f P\left(Z_T^f > 1 + \lambda | Z_t = z\right)\right\}$, is

$$\boldsymbol{f}_t^\pi(z; \lambda, \boldsymbol{\pi}_t) = \boldsymbol{\pi}_t + \left[\frac{\boldsymbol{\pi}_t^* - \boldsymbol{\pi}_t}{\sqrt{\int_t^T \gamma(s; \boldsymbol{\pi}_s)ds}}\right]\left(\frac{1+\lambda}{z}\right)\phi\left(\Phi^{-1}\left(\frac{z}{1+\lambda}\right)\right) \qquad (9.6)$$

where the function $\gamma(\cdot; \cdot)$ is defined by

$$\gamma(t; \boldsymbol{\pi}_t) := (\boldsymbol{\pi}_t^* - \boldsymbol{\pi}_t)' \boldsymbol{\Sigma}(t)(\boldsymbol{\pi}_t^* - \boldsymbol{\pi}_t) \qquad (9.7)$$

where $\boldsymbol{\pi}_t^* = \boldsymbol{\Sigma}^{-1}(t)(\boldsymbol{\mu}(t) - r(t)\boldsymbol{1})$.

Proof : Let $\hat{g}_t = \boldsymbol{f}_t - \boldsymbol{\pi}_t$, with components $\hat{g}_t^{(i)} = f_t^{(i)} - \pi_t^{(i)}$, and let $\hat{\boldsymbol{\mu}}(t)$ be defined by

$$\hat{\boldsymbol{\mu}}(t) = \boldsymbol{\mu}(t) - \boldsymbol{\Sigma}(t)\boldsymbol{\pi}_t . \qquad (9.8)$$

Then it is clear that we have

$$dZ_t^f = Z_t^f\left(\hat{\boldsymbol{g}}_t\left(\hat{\boldsymbol{\mu}}(t) - r(t)\boldsymbol{1}\right)dt + \sum_{i=1}^n\sum_{j=1}^n \hat{g}_t^{(i)}\sigma_{ij}(t)dW_t^{(j)}\right)$$

which, with $\hat{\boldsymbol{\mu}}(t)$ replacing $\boldsymbol{\mu}(t)$, is equivalent to the "proportional control" parameterization of Y_t^g of (6.14) considered previously, with $\boldsymbol{g}_t = Z_t^f \hat{\boldsymbol{g}}_t$. As such its value function is $U(t, z : 1 + \lambda)$ of (6.16), with $\hat{\boldsymbol{\mu}}(t)$ replacing $\boldsymbol{\mu}(t)$, which is equivalent to (9.5). Similarly, the optimal control is then given by $\hat{\boldsymbol{g}}_t^* = \boldsymbol{g}_t^*(z; 1 + \lambda)/z$, where $\boldsymbol{g}_t^*(\cdot; \cdot)$ is given by (6.17), with $\hat{\boldsymbol{\mu}}(t)$ replacing $\boldsymbol{\mu}(t)$. Recalling now that $\boldsymbol{f}_t^\pi = \boldsymbol{\pi}_t + \hat{\boldsymbol{g}}_t^*$, and substituting back for $\hat{\boldsymbol{\mu}}(t)$ gives the control in (9.6). ∎

9.3 Controlling for Downside Risk

The previous development gives a policy under which the maximal probability of beating the benchmark by $\lambda\%$ is $F(0, z_0; \lambda, T) = \Phi\left(\Phi^{-1}\left(\frac{z_0}{1+\lambda}\right) + \sqrt{\int_0^T \gamma(s; \boldsymbol{\pi}_s)ds}\right)$. However, as noted earlier, the nature of this policy is such that we also have the *downside risk* that the ratio at the terminal time might end up at 0, with $P(Z_T^* = 0) = 1 - F(0, z_0; \lambda, T)$, and so bankruptcy is possible. This probably entails too much risk-taking for most applications. Instead, consider then the following objective: the portfolio manager's goal is to beat the prescribed benchmark by $\lambda\%$, with the additional proviso that *the portfolio never underperform the benchmark by another given percentage, say $\delta\%$*. Formally, the portfolio manager's objective then is to determine the optimal policy $\boldsymbol{f}^\delta$ that solves

$$\sup_{\boldsymbol{f}} P\left(Z_T^f \geq 1 + \lambda, \inf_{0 \leq s \leq T} Z_s^f \geq 1 - \delta\right) . \qquad (9.9)$$

Denote this optimal value function by $\hat{F}(t, z; \lambda, \delta, T)$. The change of variable $\tilde{Z}_t^f := Z_t^f - (1 - \delta)$, then shows that

$$
\begin{aligned}
\hat{F}(t, z; \lambda, \delta, T) &:= \sup_f P\left(Z_T^f \geq 1 + \lambda, \inf_{t \leq s \leq T} Z_s^f \geq 1 - \delta | Z_t = z\right) \\
&\equiv \sup_f P\left(\tilde{Z}_T^f > \lambda + \delta, \inf_{t \leq s \leq T} \tilde{Z}_s^f \geq 0 | \tilde{Z}_t = z - (1 - \delta)\right).
\end{aligned}
$$

But this last term is precisely the problem considered earlier in Theorem 9.1., where the nonnegativity condition was implicit. As such, we know that $\hat{F}(t, z; \lambda, \delta, T) = F(t, z - (1 - \delta); \lambda + \delta - 1, T)$, where $F(t, \bullet; \bullet; T)$ was defined in (9.5). Explicitly, we have

$$
\hat{F}(t, z; \lambda, \delta, T) = \Phi\left(\Phi^{-1}\left(\frac{z - (1 - \delta)}{\lambda + \delta}\right) + \sqrt{\int_t^T \gamma(s; \pi_s) ds}\right). \tag{9.10}
$$

Similarly, the optimal policy for this problem is therefore given by

$$
f_t^\delta(z; \lambda, \delta, \pi_t) = \pi_t + \left[\frac{\pi_t^* - \pi_t}{\sqrt{\int_t^T \gamma(s; \pi_s) ds}}\right]\left(\frac{\lambda + \delta}{z - (1 - \delta)}\right)\phi\left(\Phi^{-1}\left(\frac{z - (1 - \delta)}{\lambda + \delta}\right)\right). \tag{9.11}
$$

It is interesting to note that this policy invests *more* (for a fixed z) in the risky stocks than does f_t^π of (9.6). This follows from the fact that $\phi\left(\Phi^{-1}(u)\right)/u$ is decreasing in u, and for $z \leq 1 + \lambda$, we have $(z - (1 - \delta))/(\lambda + \delta) < z/(1 + \lambda)$, for all δ.

9.4 Can we beat the Optimal Growth Policy ?

Suppose now that π_t is the *optimal growth policy*, π_t^* given by

$$
\pi_t^* := \Sigma^{-1}(t)\left(\mu(t) - r(t)\mathbf{1}\right) \equiv \left(\sigma(t)'\right)^{-1}\theta(t). \tag{9.12}
$$

As noted earleir, this policy maximizes the "growth rate from investment", see e.g., Karatzas [14, Section 9.6], and is also sometimes referred to as the *market portfolio* (from CAPM considerations).

When we place π_t^* back into the evolutionary stochastic differential equation for Z^f, and simplify, we find that $\{Z^f(\pi^*)_t\}$ satisfies

$$
dZ_t^f \equiv Z_t^f \left(f_t' - \pi_t^*\right)' \sigma(t) dW_t \tag{9.13}
$$

implying that Z_t^f is a nonnegative local martingale and hence a *supermartingale* for every admissible policy f_t. In fact, since Z^f is bounded above by $1 + \lambda$ in our application, it is indeed a martingale (although in general we would need to restrict our attention to policies that satisfy the Novikov condition (i.e., $E\left(\exp\left\{(1/2)\int_0^t f_s' f_s ds\right\}\right) < \infty$, for all t) to ensure that Z is a martingale).

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Since Z^f is a martingale we have $E\left(Z_t^f | \mathcal{F}_s\right) = Z_s^f$ for all policies f_t. Thus, since Z^f is now a "*fair game*", it is not clear if we can find an optimal strategy to beat it in finite-time. (The results of Heath and Kulldorff apply only to *favorable* games, i.e., where the underlying drift is positive. The finite-time control of *subfair* games is treated somewhat in Sudderth and Weerasinge [26], but their results are specific to a class of *strictly* subfair games, and are not applicable to our "fair" game.)

Note that for $\pi_t = \pi_t^*$, we have $\gamma(t; \pi_t^*) = 0$, for all t, and the value function of (9.5) evaluated at $\gamma = 0$ gives $F(t, z; \lambda, T) = \frac{z}{1+\lambda}$, which is of course consistent with the martingale inequality $P(Z_T^f > 1 + \lambda | Z_t^f = z) \leq z/(1+\lambda)$, which holds for all admissible policies f. Since the RHS of this inequality is independent of the policy, we can extend it to

$$\sup_f P\left(Z_T^f > 1 + \lambda \,|\, Z_t^f = z\right) \leq \frac{z}{1+\lambda}.$$

However, the value function F of (9.5) — treated as a function of γ — is *not continuous* at $\gamma = 0$. Moreover, the policy, f_t^π of (9.6) is *indeterminate* when $\pi_t = \pi_t^*$, since we also have $\gamma(t; \pi_t^*) = 0$. However, it turns out that this policy does have a limit as $\pi_t \to \pi_t^*$.

To see this, take $\pi_t = (1 - \epsilon)\pi_t^*$ in (9.6). When we do this we find that it reduces to

$$f_t^\pi(z; \lambda, (1-\epsilon)\pi_t^*) = (1-\epsilon)\pi_t^* + \left[\frac{\pi_t^*}{\sqrt{\int_t^T \theta(s)'\theta(s)ds}}\right] \left(\frac{1+\lambda}{z}\right) \phi\left(\Phi^{-1}\left(\frac{z}{1+\lambda}\right)\right).$$

This control is a continuous function of ϵ, and so we may evaluate it at $\epsilon = 0$ to obtain

$$f_t^\pi(z; \lambda, \pi_t^*) = \pi_t^* + \left[\frac{\pi_t^*}{\sqrt{\int_t^T \theta(s)'\theta(s)ds}}\right] \left(\frac{1+\lambda}{z}\right) \phi\left(\Phi^{-1}\left(\frac{z}{1+\lambda}\right)\right). \qquad (9.14)$$

The control function $f_t^\pi(z, \lambda, \pi_t^*)$ in fact *does achieve* the optimal possible value, $z/(1 + \lambda)$. We formalize this in the following.

Theorem 9.2 *Suppose that the benchmark is the optimal growth policy given by π_t^* of (9.12). Then the optimal value function is*

$$\sup_f P_{(t,z)}\left(Z_T^f > 1 + \lambda\right) = \frac{z}{1+\lambda} \qquad (9.15)$$

and this optimal value can be achieved by the policy $f_t^\pi(z, \lambda, \pi_t^*)$ in (9.14).

Remark 9.1: Note that the linear form of the value function in (9.15) precludes us from using any HJB methods.

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Proof: As noted above, since $\left\{Z_t^f, t \leq T\right\}$ is a martingale, the martingale inequality provides the upper bound $z/(1+\lambda)$. Thus the equality in (9.15) will follow if we can find a policy that achieves this upper bound. To see that $f_t^\pi(z, \lambda, \pi_t^*)$ is such a policy, replace f_t in the stochastic differential equation (9.13) with f_t^T of (9.14), and denote the resulting process by Z_t^*, and let $H_t^* := Z_t^*/(1+\lambda)$, to find

$$dH_t^* = \frac{-\pi_t^{*\prime}\sigma(t)}{\sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)ds}}\phi\left(\Phi^{-1}\left(H_t^*\right)\right)dW_t \equiv \frac{-1}{\sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)ds}}\phi\left(\Phi^{-1}\left(H_t^*\right)\right)\boldsymbol{\theta}(t)'dW_t \quad (9.16)$$

for $t \leq T$, where the last equality follows from the fact that $\pi_t^* = \sigma^{-1}(t)'\boldsymbol{\theta}(t)$.

The solution to this stochastic differential equation is

$$H_t^* = \Phi\left(\frac{\int_0^t \boldsymbol{\theta}(s)'dW_s + \sqrt{\int_0^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)ds}\ \Phi^{-1}(H_0)}{\sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)ds}}\right), \quad \text{for}\quad 0 \leq t \leq T \qquad (9.17)$$

as can be verified by an application of Ito's formula. Direct examination of (9.17) shows that H_T^* is equal to 1 or O, with

$$P(H_T^* = 1) = P\left(\int_0^T \boldsymbol{\theta}(s)'dW_s > -\sqrt{\int_0^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)ds}\ \Phi^{-1}(H_0)\right) \equiv H_0$$

where the last equality follows from the fact that $\int_0^T \boldsymbol{\theta}(s)'dW_s \sim N\left(0, \int_0^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)ds\right)$ under P. The statement of the theorem now follows since $H_t^* \equiv Z_t^*/(1+\lambda)$. ∎

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A Appendix

A.1 The Argument in Heath [11]

The modification of Heath's argument in [11] to the case treated in Section 6.2 is as follows: For Y_t^g defined by (6.14) and (6.15), let $A := \{\omega : Y_T^g \geq c\}$, and so $P(A) \leq Y_0/c$. But $P(A) \leq \sup\{P(B) : \tilde{P}(B) \leq Y_0/c\}$, and the "sup" can be evaluated via the *Neyman-Pearson Lemma*. Hence there exists a unique number λ such that $\tilde{P}\left\{\frac{dP}{d\tilde{P}} \geq \lambda\right\} = Y_0/c$. But this probability is, by (2.8), simply

$$\tilde{P}\left(\exp\left\{\int_0^T \theta(s)'d\tilde{W}_s - \frac{1}{2}\int_0^T \theta(s)'\theta(s)ds\right\} \geq \lambda\right) = Y_0/c,$$

and since under $\tilde{P}$, $\int_0^T \theta(s)'d\tilde{W}_s \sim N(0, \int_0^T \theta(s)'\theta(s)ds)$, we find

$$\ln(\lambda) = -\sqrt{\int_0^T \theta(s)'\theta(s)ds}\,\Phi^{-1}(Y_0/c) - \frac{1}{2}\int_0^T \theta(s)'\theta(s)ds$$

and hence the corresponding P probability is

$$P\left(\int_0^T \theta(s)'dW_s + \frac{1}{2}\int_0^T \theta(s)'\theta(s)ds \geq -\sqrt{\int_0^T \theta(s)'\theta(s)ds}\,\Phi^{-1}(Y_0/c) - \frac{1}{2}\int_0^T \theta(s)'\theta(s)ds\right)$$

which, by virtue of the fact that $\int_0^T \theta(s)'dW_s \sim N(0, \int_0^T \theta(s)'\theta(s)ds)$, is simply $U(0, Y_0; c)$ of (6.16). Dynamic programming then applies for arbitrary (t, y) to give $U(t, y; c)$ of (6.16).

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