Commodity Price Modelling

Lecture 2 - The Spot to Forward Relationship: Reduced-form approach

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Categories of models

1. Reduced-form pure spot/forward price models:

$$\begin{aligned} dS_t &= (r-\delta)S_t dt + \sigma S_t dW_t \\ \text{or} \quad dF(t,T) &= F(t,T)\sum_{i=1}^{n}\sigma_i(t,T)dW_t^{(i)} \end{aligned}$$

2. Fundamental Equilibrium - explicit matching of supply and demand
3. Hybrid / Structural - particularly for electricity, there are a number of models which fall between these categories.

General modelling choices often include:

- How many state variables are needed? Observable or unobservable?
- What data is available? How much to include?
- Explicit modelling of supply & demand? Discrete or continuous time?

Spot, forwards, and futures

- The spot price S_t describes the price for 'immediate' delivery.
- Forwards (and futures) are contracts for future delivery (at some $T > t$) at a fixed price $F(t,T)$ with zero cost to enter today.
- Futures are standardized exchange-traded versions of forwards, which are marked to market daily (with deterministic $r(t)$, prices equal).
- Settlement can be physical (only 1% nowadays!) or financial.
- At T the buyer receives the cashflow $S_T - F(t,T)$.
- Under the risk-neutral pricing measure $\mathbb{Q}$, we require:

$$\mathrm{e}^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}[S_T - F(t,T)] = 0 \implies F(t,T) = \mathbb{E}_t^{\mathbb{Q}}[S_T]$$

- So at T, the forward price converges to the spot price $F(T,T) = S_T$.
- A successful model for prices should ideally capture both the dynamics of S_t and those of $F(t,T)$ for all T.

Forward Curves - Delivery periods

In some energy markets, exchange-traded futures specify delivery of energy throughout an interval (eg, a month) instead at one maturity T (like swaps). Letting T_1 and T_2 equal the beginning and end of the delivery period:

- For physical delivery (throughout $[T_1, T_2]$),

$$F(t, T_1, T_2) = \mathbb{E}_t^{\mathbb{Q}}\left[\int_{T_1}^{T_2} \frac{r\mathrm{e}^{-ru}}{\mathrm{e}^{-rT_1} - \mathrm{e}^{-rT_2}} S_u du\right]$$

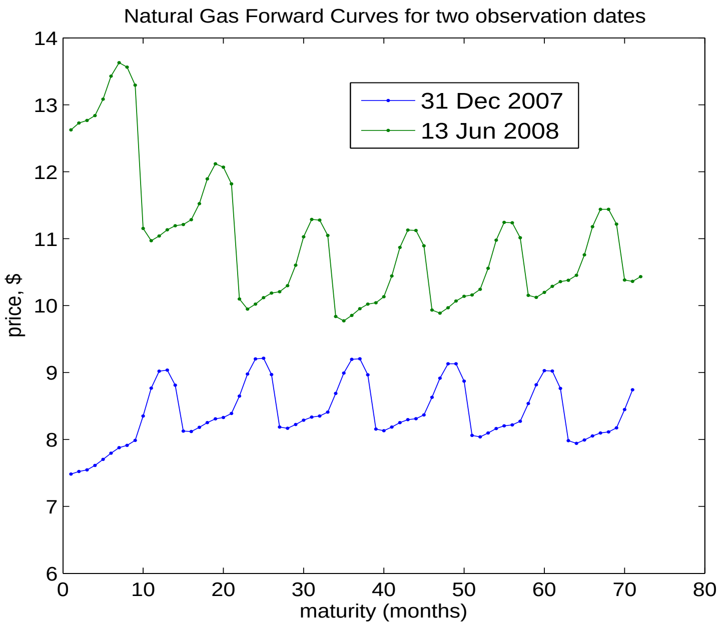
- For financial settlement (calculated and paid at T_2),

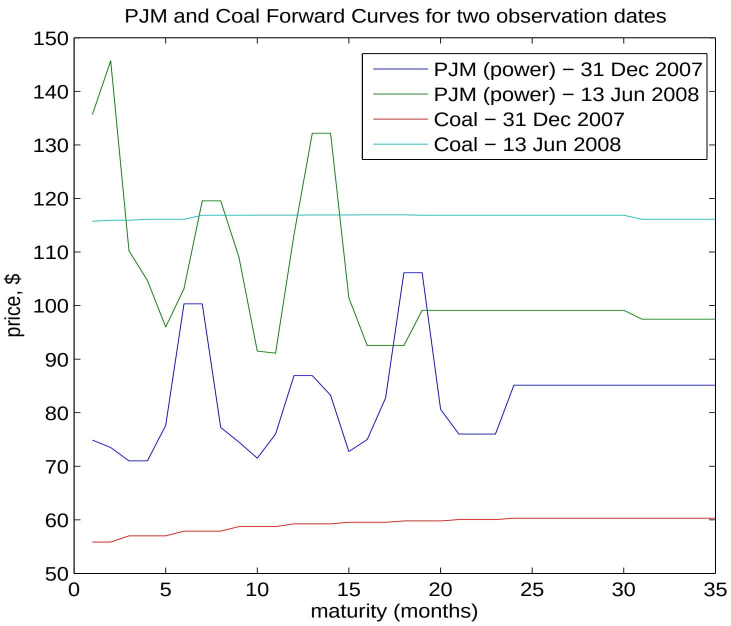
$$F(t, T_1, T_2) = \mathbb{E}_t^{\mathbb{Q}}\left[\int_{T_1}^{T_2} \frac{1}{T_2 - T_1} S_u du\right]$$

In power markets, long-term futures typically have year-long delivery periods, which then cascade into shorter periods. The modelling of a continuous forward curve $F(t, T)$ requires firstly smoothing over delivery periods.

Forward Curve Behaviour

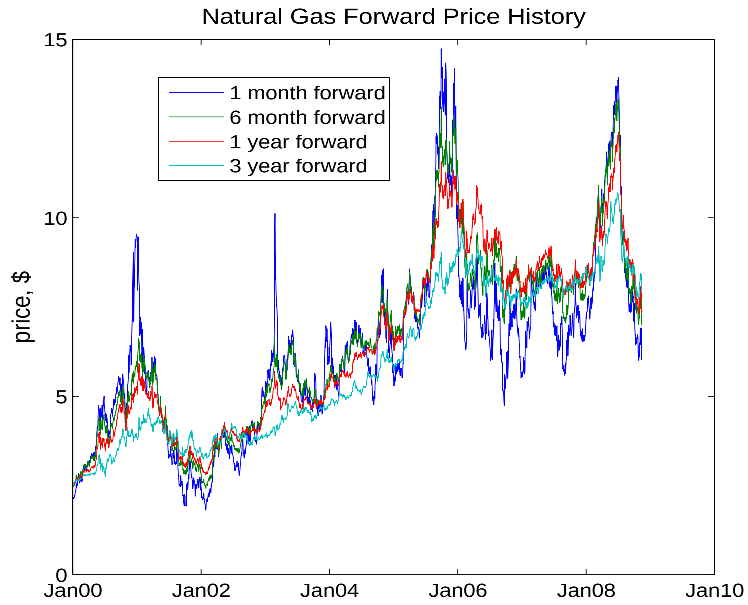
- Gas and power forward curves both show clear seasonality. A variety of different shapes (upward sloping, downward sloping, humps) are possible.
- Typically short maturity forwards move more between observation dates (though for coal there is little difference).

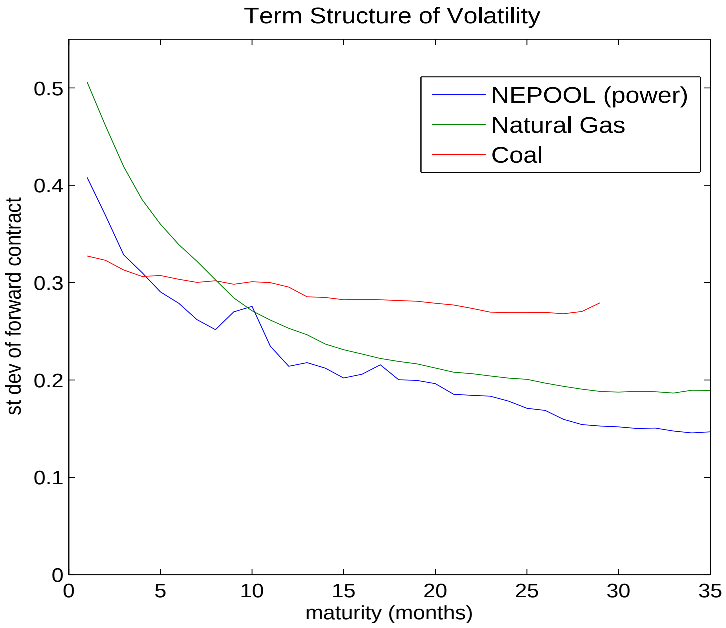




Forward Curve Behaviour

- The dynamics of different points in the forward curve are correlated but longer maturities are typically less volatile.
- A downward sloping curve is more common than upward sloping.





Forward Curves - First test of a model

Unlike futures on stocks, a variety of different shapes of forward curves can be observed in the market.

- Since storage is costly (both directly and through deterioration of stocks), we might expect upward sloping forward curves (**contango**).
- However, downward sloping (**backwardation**) has been observed in 75% of historical oil forward curves. (similarly for gas).
- Theory of Normal Backwardation: Keynes (1930) claimed backwardation is typical due to risk-averse producers hedging.
    - Key implicit assumption - hedgers are net short overall
- Most agree that forward curves provide an indication of which direction the market expects spot prices to go, but the predictive power of forwards is often weak (eg, Fama and French, 1987).
- The Theory of Storage (originating in the 1930s) led to the idea of a **convenience yield** to explain the spot to forward relationship.

The Theory of Storage

The Theory of Storage (originally proposed in 1930s):

- Holding a physical commodity provides benefits which futures don't:
  - A way of avoiding disruptions to production (weathering supply and demand shocks)
  - An "embedded timing option attached to the commodity" (Brennan 1958)
- Can be treated analogously to a dividend yield for stock prices
- Costly storage (or high interest rates) counters the effect of the convenience yield
- Famous cost of carry relationship ('no-arbitrage argument'):

$$F(t,T) = S_t \exp\{(r(t) + c(t) - \delta(t))(T - t)\}$$

where $r(t)$, $c(t)$ and $\delta(t)$ are interest rates, storage costs and convenience yield.

The Theory of Storage

- The no arbitrage argument is complicated by various factors:
  - short selling typically not possible
  - storage (inventories) can't go negative! - hence 'stock outs' break this relationship
  - for non-storable commodities, it disappears altogether.
- Often convenience yield defined net of storage costs (which change only gradually):

$$F(t,T) = S_t \exp\left\{\left(r(t) - \delta(t)\right)(T - t)\right\}$$

- Early models chose $\delta(t)$ constant or deterministic, then occasionally $\delta(S_t)$, now typically stochastic δ_t.
- Though not observable, the convenience yield intuitively is inversely related to inventory levels.

The Theory of Storage

Another key observation of commodity spot and forward prices is their dependence on inventory levels.

- Futures prices can drive storage decisions (or arguably, the other way around) as well as production and consumption decisions.
- Studies have also linked prices with estimates of oil reserves in the ground.
- Low inventory levels lead to high spot prices and increased backwardation, but also high volatility and reduced spot to forward correlation (eg, Ng and Pirrong, 1994, Fama and French, 1988, Geman and Nguyen 2002, and many others).

The convenience yield δ_t is a reduced form approach to capturing these effects.

Forward Curves - Other Features

- Samuelson's Hypothesis is the observation that the volatility of forward contracts increases as maturity approaches

$$\Longrightarrow \quad \text{suggests mean reverting spot price } S_t.$$

- However, Clewlow and Strickland (2000) emphasise that the observed volatility of very long maturity forwards does not seem to go to zero.
- Similarly, Koekebakker and Ollmar (2005) use Principal Component Analysis (PCA) to show that only 75% of the forward price variation can be explained by two factors, while this number is closer to 95% in other markets such as interest rates.

Earliest Models

- Black (1976) suggested modelling forward prices as lognormal and with a constant volatility (as in Black-Scholes).
- The basic Geometric Brownian Motion model which reproduces the cost of carry relationship is, under $\mathbb{Q}$,

$$dS_t = (r-\delta)S_t dt + \sigma S_t dW_t \implies \frac{dF(t,T)}{F(t,T)} = \sigma dW_t$$

- We have lognormal spot and forward prices and thus Black-Scholes like formulas for options (on spot or forward). eg, for a call on a forward $V(t,T_o,T_f)$ (with payoff $\max(F(T_o,T_f)-K,0)$ at T_o).

$$V(t,T_o,T_f) = \mathrm{e}^{-r(T_o-t)}\left(F(t,T_f)\Phi(d_1) - K\Phi(d_2)\right)$$

where $d_1 = \frac{\log(F(t,T_f)/K)+\frac{1}{2}\sigma^2(T_o-t)}{\sigma\sqrt{T_o-t}}$, $d_2 = d_1 - \sigma\sqrt{T_o-t}$

- Many obvious weaknesses: flat term structure of volatility, no mean reversion, single factor, very simple forward curves.

Schwartz (1997) one-factor model

Mean reversion is now well established as a feature of commodity markets

- Here we no longer maintain the cost of carry relationship (or that discounted adjusted spot prices are martingales), but simply set

$$dS_t = \kappa(\tilde{\mu} - \log S_t)S_t dt + \sigma S_t dW_t$$

  Alternatively start from standard Ornstein Uhlenbeck (OU) process:

$$dX_t = \kappa(\mu - X_t)dt + \sigma dW_t$$

  and $S_t = \exp(X_t)$ or $S_t = \exp(f(t) + X_t)$ where $f(t)$ is a deterministic seasonality function (eg, Lucia and Schwartz 2002).

- By Ito's Lemma, $\tilde{\mu} = \mu + \frac{\sigma^2}{2\kappa}$ or $\tilde{\mu} = \mu + \frac{\sigma^2}{2\kappa} + \frac{1}{\kappa}(f(t) + f^{'}(t))$
- As for Vasicek model, simplest approach to capturing risk preferences is a constant market price of risk λ. Then if you start with a mean reversion level μ under $\mathbb{P}$, you get $\mu - \lambda\sigma$ under $\mathbb{Q}$.

Schwartz (1997) one-factor model

This 'exponential OU process' ensures positive prices and is very common in commodity price modelling.

- Again forward prices are lognormal so closed-form option pricing is still easy

$$\begin{aligned} V(t,T_o,T_f) &= \mathbb{E}_t^{\mathbb{Q}}\left[\max(F(T_o,T_f)-K,0\right] \\ &= \mathrm{e}^{-r(T_o-t)}\left(F(t,T_f)N(d_1)-KN(d_2)\right) \end{aligned}$$

where $d_1 = \frac{\log(F(t,T_f)/K)+\frac{1}{2}w}{\sqrt{w}}$, $d_2 = d_1 - \sqrt{w}$ and

$$w = \int_t^{T_o} \sigma^2 \mathrm{e}^{-2\kappa(T_f-u)}du = \frac{\sigma^2}{2\kappa}\left(\mathrm{e}^{-2\kappa(T_f-T_o)} - \mathrm{e}^{-2\kappa(T_f-t)}\right)$$

- By including observed forward prices here, the model is made consistent with the forward curve.
- Forward curves dynamics: $\frac{dF(t,T)}{F(t,T)} = \sigma \mathrm{e}^{-\kappa(T-t)}dW_t$

Schwartz (1997) two-factor model

- A natural way of combining the two one-factor models discussed above is by allowing the mean reverting process to be the convenience yield

$$\begin{aligned} dS_t &= (r - \delta_t)S_t dt + \sigma_1 S_t dW_t \\ \delta_t &= \kappa(\mu - \delta_t)dt + \sigma_2 dB_t \\ dW_t dB_t &= \rho dt \end{aligned}$$

- We retain the intuition of ‘cost-of-carry’ while capturing the volatility term structure and a variety of forward curve shapes (eg, single hump).
- Futures vol is $\sqrt{\sigma_1^2 + \frac{\sigma_2^2}{\kappa}(1 - \mathrm{e}^{-\kappa(T-t)})^2 - 2\sigma_1\sigma_2\rho(1 - \mathrm{e}^{-\kappa(T-t)})}$
- Closed-from expressions for forwards and options still exist, as both spot and forward prices are still lognormal.
- Stochastic interest rates can be considered (eg, Schwartz 3-factor includes Vasicek for short rate, also see Miltersen and Schwartz, 1998) but not typically considered significant risk factors.

Alternative two-factor models

Schwartz and Smith (2000) separate dynamics into long-term and short-term components. Let $S_t = \exp(X_t + Y_t)$ or $S_t = \exp(f(t) + X_t + Y_t)$ where

$$\begin{aligned} dX_t &= -\kappa X_t dt + \sigma_1 dW_t \\ dY_t &= \mu dt + \sigma_2 dB_t \\ dW_t dB_t &= \rho dt \end{aligned}$$

Again, spot and forward prices are lognormal, and option pricing formulas are straightforward. So how do these models compare?

- In summary, they are the same! - different intuition, same dynamics.
- As pointed out in Schwartz and Smith (2000), the convenience yield two-factor model (Gibson and Schwartz, 1990) is equivalent as we can write the factors as linear combinations of factors in this model.

Alternative two-factor models

Various other two and three factor variations proposed:

- Pilipovic (1997) assumes a mean reverting spot price to a stochastic non-stationary long term level:

$$
\begin{aligned}
dS_t &= \kappa(L_t - S_t)dt + \sigma_1 S_t^{\alpha} dW_t \\
dL_t &= \mu L_t dt + \sigma_2 L_t^{\beta} dB_t
\end{aligned}
$$

  with $\alpha, \beta \in (0,1)$ (additive or proportional noise) and correlation zero.
- Possible 3-factor extensions include (eg, Lavi-Lavassani *et al*, 2001):
  - L_t mean-reverting to a non-stationary level M_t.
  - Stochastic volatility with σ_t an OU process.
- All of these similar approaches emphasize (either explicitly or implicitly) the importance of different factors for different time horizons, and thus different parts of the forward curve.
- For electricity, a natural extension is to add jumps.

One more from Schwartz

- Nielsen and Schwartz (2004) propose a modification to the convenience yield model to attempt to fill in the missing features.
- The volatility of S_t is allowed to depend on the convenience yield δ_t. Specifically, $\sigma_t = \sqrt{\beta_1 \delta_t + \beta_2}$ and the volatility of δ_t is similar.
- Forward and option prices still available as we are still in the 'exponential affine' framework (ie, Duffie, Pan, Singleton, 2000).
- This extension addresses the many studies linking inventory with price AND volatility (eg, Geman and Nguyen 2002), as well as linking degree of backwardation with volatility (eg, Ng and Pirrong, 1994).
- Other observations - inventory levels also impact the spot to forward correlation, which is harder to capture.

What are the alternative approaches?

Reduced-form forward curve models:

- Instead of deriving the volatility function in $\frac{dF(t,T)}{F(t,T)} = \sigma(t,T)dW_t$ from S_t, we start with a choice for the function $\sigma(t,T)$ (like HJM).
- If possible, determine the spot price dynamics S_t.

Equilibrium models ('economics style' approaches):

- Specify the supply and demand functions explicitly, as well as inventory levels
- Use optimisation approach to determine the optimal storage and production decisions

Key References

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